Non-Oberbeck-Boussinesq effects in turbulent thermal convection in ethane close to the critical point

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As shown in earlier work (Ahlers et al., J. Fluid Mech. 569, 409 (2006)), non-Oberbeck Boussinesq (NOB) corrections to the center temperature in turbulent Rayleigh-Bénard convection in water and also in glycerol are governed by the temperature dependences of the kinematic viscosity and the thermal diffusion coefficient. If the working fluid is ethane close to the critical point the origin of non-Oberbeck-Boussinesq corrections is very different, as will be shown in the present paper. Namely, the main origin of NOB corrections then lies in the strong temperature dependence of the isobaric thermal expansion coefficient \(\beta(T)\). More precisely, it is the nonlinear \(T\)-dependence of the density \(\rho(T)\) in the buoyancy force which causes another type of NOB effect. We demonstrate that through a combination of experimental, numerical, and theoretical work, the latter in the framework of the extended Prandtl-Blasius boundary layer theory developed by Ahlers et al., J. Fluid Mech. 569, 409 (2006). The latter comes to its limits, if the temperature dependence of the thermal expansion coefficient \(\beta(T)\) is significant. The new measurements reported here cover the ranges \(2.1 \lesssim Pr \lesssim 3.9\) and \(5 \times 10^9 \lesssim Ra \lesssim 2 \times 10^{12}\) and are for cylindrical samples of aspect ratios 1.0 and 0.5.

I. INTRODUCTION

Fluid motion in the presence of temperature gradients is an important phenomenon in nature and industrial processes. Among the many examples are oceanic streams, cloud motions, and gusts of wind that can be felt on a human scale. The paradigmatic system for such thermally driven flows is the Rayleigh-Bénard (RB) setup: a fluid-filled container heated from below and cooled from above. In this classical problem, the flow is determined by the scale and geometry of the container, the material properties of the working fluid, and the top-down temperature difference \(\Delta \equiv T_c - T_f > 0\). In the last two decades, considerable progress has been achieved in our understanding of global and local properties and the flow organization of turbulent RB convection, through a combination of experimental (see e.g. [1–24]), numerical (see e.g. [25–35]), and theoretical work (see e.g. [36–38]).

The temperature difference in a RB cell can be increased in a controlled way. However, in principle the transport coefficients of the fluid can depend on the local temperature and density and thus vary across the height \(L\) of the container. Since space-dependent properties of such kind are undesirable in first instance, one tends to restrict the convection regime to sufficiently small intervals of \(\Delta\). But even so, further simplifications are progressively required in the analysis of RB convection. In this spirit, a standard approximation due to Oberbeck [39] and Boussinesq [40] assumes that (see also [41, 42]):

**OB.1** The dynamic viscosity \(\eta\), the thermal conductivity \(\Lambda\), the thermal expansivity \(\beta\), and the isobaric specific heat \(c_P\) are constant throughout the fluid.

**OB.2** Density variations are taken into account only in the buoyancy force term.

**OB.3** The temperature dependence of the density \(\rho\) is linearized in the buoyancy force as:

\[
\rho(T) = \rho_m - \rho_m \beta_m (T - T_m),
\]

where \(T_m \equiv (T_c + T_f)/2\) is the arithmetic mean temperature between the plates and \(X_m = X(T_m)\) denotes the fluid property \(X\) evaluated at \(T_m\).

Next to the aspect ratio, within the OB approximation two dimensionless parameters characterize the RB flow: The Prandtl number \(Pr \equiv \nu_m/\kappa_m\) follows from the ratio of the kinematic viscosity \(\nu_m \equiv \eta_m/\rho_m\) and the thermal diffusivity \(\kappa_m \equiv \Lambda_m/\rho_m c_p \rho m\). The dimensionless thermal driving can be conveniently represented by the Rayleigh number \(Ra \equiv \beta_m g L^3 \Delta/(\nu_m \kappa_m)\), where \(g\) denotes the gravitational acceleration.

The manner in which high Rayleigh numbers are achieved is crucial for the emergence of non-Oberbeck-Boussinesq effects (NOB). Since turbulent convection may involve spatiotemporal changes in the fluid properties, considerable efforts have been devoted to the identification of dominating sources of NOB effects. In liquids like water [43] and glycerol [44], [45], for example, NOB effects are dominated by deviations from (OB.1) since the viscosity strongly decreases with temperature. On the other hand, when the working fluid is gaseous ethane [46], deviations from (OB.1) and (OB.2) lead to NOB effects stronger than those in the aforementioned liquids.
In the present study, we shall focus on deviations from (OB.3) by considering the nonlinear temperature dependence of the buoyancy force. In particular, ethane close to its critical point [47] is chosen as the working fluid and the temperature $T_c$ in the center of the container is measured as indicator of NOB effects.

There are two possibilities to characterize the physics beyond condition OB.3 as described by eq. (1), if the density $\rho(T)$ has a strong nonlinear $T$-dependence. First, one can introduce a $T$-dependent thermal expansion function $\beta(T)$ instead of $\beta_m$, defined in terms of the density $\rho(T)$ by

$$\rho(T) \equiv \rho_m - \rho_m(T - T_m)\beta(T).$$

(2)

Second, one refers to the common isobaric thermal expansion coefficient $\beta(T)$, defined as usual by

$$\beta(T) \equiv -\frac{1}{\rho(T)} \frac{\partial \rho(T)}{\partial T},$$

(3)

which now is temperature dependent. Both are related by $\beta(T) = -\partial \log[1 - (T - T_m)\beta(T)] / \partial T$. Under the condition OB.3 of linear $T$-dependence of $\rho$ the thermal expansion function is constant, $\beta = \beta_m$, while the expansion coefficient $\beta(T)$ is given by $\beta(T) = \beta_m/[1 - (T - T_m)\beta_m]$, still depending on temperature. Of course all three coincide at $T_m$. An advantage of considering the thermal expansion coefficient $\beta(T)$ is that it is a well defined thermodynamic derivative. The advantage of $\beta(T)$ on the other hand is that it immediately reflects the nonlinear $T$-dependence of $\rho(T)$. Also $\beta(T)$ refers to a single thermodynamic state and describes the local $T$-slope on an isobar (normalized by the local density), while $\beta(T)$ refers to a pair of states, namely to the reference state $T_m$ in addition to $T$ and describes the secant to the $\rho(T)$-curve (normalized by the reference density). $\beta(T)$ will therefore in general vary less with $T$ than $\beta(T)$.

It will turn out that it is the significantly different $T$-dependence of $\beta(T)$ (or $\beta(T)$) on the two sides of the critical isochore of ethane, which leads to opposite shifts of the center (bulk) temperature $T_c$, yielding $T_c < T_m$ on the gas-like (i.e., high-temperature, see Fig. 1 below) side and $T_c > T_m$ on the liquid-like (low-temperature) side. On the gas-like side $\beta(T)$ increases from bottom to top and on the liquid-like side it decreases.

Our approach consists of three stages: boundary layer (BL) theory, experiments, and direct numerical simulations (DNS). First, we address in section II an extension of boundary layer theory that considers deviations from (OB.1) and (OB.2). Even though the buoyancy force is not included in the BL equations (only the longitudinal momentum is taken into account here), we compute $T_c(\Delta)$ for several pressures $P_m$. Then, experimental measurements of $T_c(\Delta)$ are presented in section III B and compared with BL results in section IV. Given the significant discrepancies between part of them, we address in section V direct numerical simulations that explicitly consider deviations from (OB.3). In particular, for $T_m = 27^\circ$ C and $P_m = 51.72$ bar, it is shown that NOB effects in ethane are dominated by the nonlinear dependence of the buoyancy force on temperature. Finally, our conclusions are summarized in section VI. Appendices A and B are devoted to the derivation of the boundary layer equations with variable transport coefficients and Appendix C compiles the Nusselt number corrections for the real and various hypothetical ethane-like fluids in a table.

II. BOUNDARY-LAYER THEORY

A central aspect in Rayleigh-Bénard convection is the understanding of the boundary layers formed along the bottom and top plates. Though they preserve a laminar character for $Ra \leq 10^{12}$, their instabilities impact the Nusselt number $Nu$ (the effective heat flux relative to thermal conduction $\Lambda_m \Delta/L$) [48, 49]. As reported in references [43, 44, 46], BL flows of this nature are significantly influenced by the coupling between the fluid properties and the temperature gradient across the container. In particular, it was shown that NOB effects on $T_c$ can be reasonably described by extending the Prandtl-Blasius boundary layer theory [50, 51]. Next we review such an extension and further assess its intrinsic limitations.

Assume that the density $\rho$, the temperature $T$, and the velocity $u$ are stationary fields, which depend only on the longitudinal $x$ and transverse $z$ coordinates. Then, under the boundary-layer approximation, we write the continuity and the $x$-momentum equations as (see also appendix A 1):

$$\frac{\partial}{\partial x} (\rho u_x) + \frac{\partial}{\partial z} (\rho u_z) = 0,$$

(4)

$$\rho \left\{ u_x \frac{\partial u_x}{\partial x} + u_z \frac{\partial u_z}{\partial z} \right\} = \frac{\partial}{\partial z} \left\{ \eta \frac{\partial u_x}{\partial z} \right\}. $$

(5)

Here $z$ measures the vertical distance from the bottom or top plates, respectively, the velocity components at $z = 0$ are subject to no-slip boundary conditions: $u_x(x, 0) = 0$ and $u_z(x, 0) = 0$. Moreover, in the bulk of the flow, $u_z$ is supposed to match the large scale wind velocity $U_z$ in the center (bulk) of the RB sample [52], i.e., $u_x(x, \infty) = U_c$. Note that within the BL theory we cannot calculate $U_z$; here we only have to assume that it is the same close to the top and the bottom BL which is supported by our numerical simulations reported in section V.

In the same spirit, the temperature field $T(x, z)$ is governed by (cf. appendix A 2):

$$\frac{\gamma}{\rho c_p} \frac{\partial}{\partial z} \left\{ \Delta \frac{\partial T}{\partial z} \right\} = u_x \frac{\partial T}{\partial x} + u_z \frac{\partial T}{\partial z} + \frac{\gamma - 1}{\beta} \left\{ \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right\}, $$

(6)

where $\gamma \equiv c_p/c_V$ is the ratio between the isobaric and isochoric specific heats and $\beta = -\rho^{-1} \partial \rho/\partial T$ de-
notes the isobaric thermal expansion coefficient. At the 
plates \( T(x, 0) = T_{b,t} \) and in the bulk (center) of the flow 
\( T(x, \infty) = T_c \).

The coupling between the bottom and top boundary 
layers is determined (cf. [43]) by the heat fluxes \( Q_{b,t} \) 
through the plates, considered to be equal:

\[
Q_b = -\Lambda_b \frac{\partial T}{\partial z}|_b = -\Lambda_t \frac{\partial T}{\partial z}|_t = Q_t. \tag{7}
\]

This condition establishes an implicit dependence of the 
center temperature \( T_c \) on the heat fluxes \( Q_b = Q_t = Q \). 
Note again that both the dynamic viscosity \( \eta \) and the 
heat conductivity \( \Lambda \) depend on both temperature and 
density, i.e., \( \eta(T, \rho) \) and \( \Lambda(T, \rho) \). Before addressing 
the technicalities around the (numerical) integration of equa-
tions (4)–(7), we shall benefit from a key argument in 
boundary-layer theory: Prandtl’s self-similar ansatz.

A. Self-similarity

Because of the \( y \)-independence, assumed in the Prandtl 
BL theory, the boundary layer flow is mathematically a 
2D flow. Therefore the system of partial differential equa-
tions (4)–(6) for the BL flow can be reduced to ordinary 
differential equations (ODEs) by introducing a stream 
function \( \Psi \). We do this differently from the usual pro-
cedure by including in its definition the density in order to 
automatically fulfill the continuity equation by construc-
tion.

\[
\hat{\rho} u_x = \frac{\partial \hat{\Psi}}{\partial x}, \quad \hat{\rho} u_z = -\frac{\partial \hat{\Psi}}{\partial z}, \tag{8}
\]

where \( \hat{\rho} \equiv \rho/\rho_m \) is the density nondimensionalized with 
\( \rho_m = \rho(T_m, P_m) \). Apparently the continuity equation au-
tomatically follows from (8)–(9). Next, cf. Appendix B, 
we may introduce a self-similarity variable \( \tilde{Z} \equiv z/\ell_c(x) \) 
and a similarity function \( \tilde{\Psi}(\tilde{Z}) = \Psi(x, z)/(\ell_c(x) U_c) \), 
such that \( \ell_c(x) = \sqrt{xu_m/U_c} \). Thus the velocity com-
ponents are

\[
\hat{u}_x = U_c \frac{\tilde{\Psi}'}{\hat{\rho}}, \quad \hat{u}_z = \frac{\nu_m}{2\ell_c} \left\{ \frac{1}{2} \frac{\tilde{\Psi}''}{\hat{\rho}} - \frac{\tilde{\Psi}'}{\hat{\rho}} \right\} \tag{10}
\]

with boundary conditions \( \hat{\Psi}'(0) = 0 = \tilde{\Psi}'(0) \) and 
\( \tilde{\Psi}'(\infty) = \hat{\rho}_c \).

In terms of (10), the viscous BL equation (5) can be 
written as:

\[
0 = \hat{\eta} \tilde{\Psi}'' + \left\{ \frac{1}{2} \frac{\tilde{\Psi}'}{\hat{\rho}} + \hat{\rho}' \right\} \tilde{\Psi}' + \left\{ -\frac{1}{2} \hat{\rho}' \tilde{\Psi}' + \left[ \frac{1}{2} \left( \frac{\hat{\rho}}{\hat{\rho}_c} \right)^2 - \frac{\hat{\rho}''}{\hat{\rho}} \right] \hat{\eta} \frac{\hat{\rho}'}{\hat{\rho}} \right\} \tilde{\Psi} \tag{11}
\]

Here \( \hat{\eta} \equiv \eta/\eta_m \) is the dimensionless viscosity, whose \( \tilde{Z} \)-
dependence \( \hat{\eta}' \) is given by

\[
\hat{\eta}' = \left( \frac{\partial \hat{\eta}}{\partial \hat{\Theta}} \right) \hat{\Theta}' + \left( \frac{\partial \hat{\eta}}{\partial \hat{\rho}} \right) \hat{\rho}',
\]

where \( \hat{\Theta} \equiv (T - T_1)/\Delta \) denotes the dimensionless 
temperature.

Next, assuming that the pressure \( P_m \) is constant 
throughout the fluid, one finds

\[
\hat{\rho}' = -\hat{\rho} \hat{\beta} \hat{\Theta}', \tag{12}
\]

with \( \hat{\beta} \equiv \beta \Delta \). The boundary conditions at the respective 
walls are \( \hat{\rho}(0) = \hat{\rho}_{b,t} \), \( \hat{\rho}'(0) = -\hat{\rho}_{b,t} \hat{\beta}_{b,t} \hat{\Theta}'_{b,t} \), and 
\( \hat{\rho}(\infty) = \hat{\rho}_c \).

Finally, we also write the temperature equation (6) in 
self-similar form as (see Appendix B2)

\[
\hat{\Lambda} \hat{\Theta}'' + \left\{ \frac{1}{2} \hat{c}_P \hat{\Pr} \hat{\Psi} + \hat{\Lambda} \hat{\Psi}' \right\} \hat{\Theta}' = 0, \tag{13}
\]

where \( \hat{\Lambda} \equiv \Lambda/\Lambda_m \) and \( \hat{c}_P \equiv c_P/c_{P,m} \). Equation (13) is 
subject to \( \hat{\Theta}(0) = \hat{\Theta}_{b,t} \) and \( \hat{\Theta}(\infty) = \hat{\Theta}_c \).

B. Results

The coupled ODEs (11)–(13) with the respective 
boundary conditions and the heat-flux conservation (7) 
are solved numerically with a shooting method [53]. The 
integration domain is restricted to \( \Delta \)-intervals where the 
transport properties are concave/convex functions of the 
temperature. In particular, we have chosen ethane as the 
working fluid since its properties are known very well 
[47], even close to its critical point \( (T_*, P_*, \rho_*) \) [see figure 
1]. All material properties \( \eta, \Lambda, \rho, \beta, \) and \( c_P \) are implemented 
in their full dependence on \( T \) in this manner, the 
computation of temperature and density profiles does not 
involve any fit parameter.

1. Vertical profiles

An insight into the structure of the BLs can be achieved 
by studying typical profiles along the \( z \)-direction. To 
describe them, let us consider a representa-
tive case in which the pressure is fixed at \( P_m = 0.849 \cdot P_c \), 
the mean temperature at \( T_m = 40 \, ^{\circ} \mathrm{C} \), and the thermal 
difference between the plates at \( \Delta = 15 \, \mathrm{K} \).

In figure 2, the temperature \( \hat{\Theta} \) and density \( \hat{\rho} \) are 
plotted as functions of the similarity variable \( \tilde{Z} \). As shown 
in panel 2a, the center temperature \( \hat{\Theta}_c \) is smaller 
than the mean temperature \( \hat{\Theta}_m = 0.5 \), clearly indicating a 
top-down symmetry breaking. Such symmetry breaking 
is also reflected in the density profiles shown in panel 
2b, since the center density \( \hat{\rho}_c \) is larger than the mean 
density \( \hat{\rho}_m = 1 \). Notwithstanding the pronounced curva-
tures in \( \hat{\Theta}(\tilde{Z}) \) and \( \hat{\rho}(\tilde{Z}) \), we shall restrict our attention
FIG. 1: Pressure-temperature plane in reduced units. Star: critical point of ethane ($T_\ast = 32.18^\circ\text{C}, P_\ast = 48.718\text{ bar}$). Heavy line: liquid-vapor coexistence curve. Dotted line: critical isochore. The horizontal arrows show the maximum temperature intervals $\Delta$, centered at $T_m = 27^\circ\text{C}$ (left) and $T_m = 40^\circ\text{C}$ (right). The pressures are $P_m/P_\ast = 0.849, 0.991, 1.026, 1.062$, and $1.104$ (bottom to top).

To the asymptotic value $\tilde{\Theta}_c$ as a convenient indicator of NOB effects.

2. Center temperature

To compute the difference $T_c - T_m$ as a function of $\Delta$, we have chosen a particular set of isobars in the phase diagram of ethane. As shown in figure 1, our selection of $\Delta$-intervals falls into two classes: (i) those intervals centered at $T_m = 27^\circ\text{C}$ and (ii) those centered at $T_m = 40^\circ\text{C}$.

As for the latter (the more gaseous case), figure 4a shows that the center temperature is a decreasing function of $\Delta$. The top-down symmetry of the flow is broken in such a way that the top boundary layer tends to become thinner than its bottom counterpart, eventually leading to a temperature reduction in the center of the flow. Though this result has been originally reported and explained already in reference [46], we briefly mention it here for completeness of discussion.

Focussing now on the class of $\Delta$-intervals centered at $T_m = 27^\circ\text{C}$, figure 4b shows that the center temperature becomes larger than the mean temperature between the plates. Such NOB effect is different to what we found in reference [46] and repeated in figure 4a, where we have focused on the more gas-like case.

To understand this we argue that the intervals under consideration (centered at $T_m = 27^\circ\text{C}$) now correspond to a region of the phase diagram, where the material properties of ethane behave more similar to those of the liquid phase. NOB effects in classical liquids (such as water and glycerol) were already discussed in references [43–45]. One of our aims in the present work is to further assess the differences in the NOB effects between the more liquid-like versus the more gas-like fluids, see Sec. III. It will turn out that this will show us the limitations of boundary-layer theory, see Sec. IV. To this end, we first consider now additional experimental details on the $\Delta$ dependence of the center temperature $T_c(\Delta)$.
III. EXPERIMENT

A. Apparatus

The apparatus was described in detail before in Ref. [54], where a schematic diagram is shown in Fig. 2. Here we give a brief description and details specific to the present high-pressure sample cell shown in Fig. 3. Working from the inside out, the sample cell was surrounded by a can containing ambient air. The air space inside the can was filled with low-density open-pore foam to prevent convection outside the sample. The maximum possible diameter of the sample top plate was 10 cm, allowing for inside sample diameters typically up to about \( D = 8 \) cm. The entire apparatus was of sufficient length to accommodate a sample with \( L \simeq 16 \) cm (\( \Gamma \simeq 0.5 \)). Heat was applied at the sample bottom by a metal-film heater covering the entire active bottom-plate area uniformly.

The top plate was cooled by a circulating water bath. The water was cooled when passing through a heat exchanger external to the main apparatus which in turn was cooled by a separate water circuit driven by a Neslab or Lauda refrigerated circulator with a temperature stability of 0.01°C. Just before entering the apparatus, the water was heated by a heater consisting of about 25 m of teflon-insulated AWG30 (0.5 mm dia) copper wire, stuffed into the inlet line and thus immersed in the water. The large contact area between the water and the heater wire provided excellent heat exchange and uniform heating of the water. The heater was computer controlled in a feedback loop with a thermometer located in the top plate of the sample cell. The bath-temperature stability achieved in this way was a few tenths of a milli-Kelvin.

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The water entered the bottom center of the apparatus, flowed upward through an annular channel around the can, and was distributed over the top plate by a set of jets. With this arrangement the entire can was kept at the top-plate temperature and parasitic heat loss from the side wall and the bottom plate due to conduction through the air/foam as well as by radiation was reduced to a level that was negligible compared to the heat transport by the convecting fluid.

After cooling the top plate the water returned through an annular channel located just outside of and mildly insulated from the incoming channel. Since the water, while cooling the top plate, was never heated by more than a few mK, the returning water provided an excellent adiabatic thermal shield at the top-plate temperature, thus stabilizing the interior temperatures and preventing significant variations in time of the parasitic heat losses from the bottom plate. The entire apparatus sat on a chlorinated poly (vinyl chloride) (CPVC) base plate with appropriate channels and feed-throughs to accommodate the water circuit, the electrical leads, and the fill capillary going to the sample.

One of two high-pressure sample-cells (see Fig. 3) was installed inside the can. It had the shape of a cylinder with \( D = 7.63 \) cm. One cell had an internal length \( L = 7.62 \) cm, corresponding to an aspect ratio \( \Gamma \equiv D/L = 1.00 \). Another one had \( L = 15.24 \) cm, yielding \( \Gamma = 0.500 \). The top and bottom plates consisted of thick copper disks. Each of the two plates had an anvil, 1.59 cm thick, of diameter essentially equal to \( D \), that was a close slide fit in the type 4340 steel side wall. After machining, the side wall was heat treated at 830°C and oil quenched. This procedure is expected to lead to a tensile strength of about 13 kbars. The side wall had a flange at each end, of thickness 0.95 cm. Each flange was bolted (not shown in the figure) and “O”-ring-sealed to one of the copper end plates. A top and bottom thin section of the side wall had a thickness of 0.051 (0.076) cm for the \( \Gamma = 1 \) (\( \Gamma = 0.5 \)) cell. This thin section overlapped the copper anvils and extended into the sample region by 0.95 cm. Connecting the thin section was a central section of wall thickness 0.15 cm that provided enhanced strength; since the turbulent system contained only a very small thermal gradient in its center, the thicker wall section did not significantly enhance the wall heat-transport. The sample entered the bottom copper plate through a capillary from the side, and then proceeded through a very small hole (shown in white on the left side of the figure) into the gap between the bottom-plate anvil and the side wall.

The sample was connected to a manifold through a capillary. Also connected to the manifold was a separate pressure-regulation volume of 600 (1000) cm\(^3\) in the case of the \( \Gamma = 1 \) (\( \Gamma = 0.5 \)) cell that could be heated above the ambient temperature by a heater wrapped around its outside. The temperature of this “hot volume” was controlled in a feedback loop with a pressure gage [55]. The pressure stability typically was better that one milli-bar. The entire system was designed to safely withstand pressures up to 60 bars. For the pressure measurements we used a Paroscientific model 745 pressure standard with an accuracy or 80 ppm (about 6 milli-bars) and a reso-
A substantial fraction of the heat current passed through the side wall. This current was measured for the evacuated cell and subtracted from all other measurements; but as was recognized some time ago ([56, 57]), this is not an adequate procedure because of the height-dependent temperature gradients that prevail in the wall when the cell contains turbulently convecting fluid. We did not attempt a correction for this non-linear side-wall effect in the present case because we do not believe that a reliable correction is possible when the side-wall conductance is large. For this reason our values of $N_u$ under OB condition are about 25% larger than other measurements at similar Prandtl numbers ([16]). However, we believe that the deviations of $N_u$ and of $T_c$ from their Boussinesq values, which (as we shall see below in Sect. III B 2) depend primarily on the nature of the top and bottom boundary layers rather than on the fluid interior, were obtained reliably.

The top and bottom temperatures $T_t$ and $T_b$ were determined from the average of six thermistors imbedded close to the fluid in each of the top and bottom plates ([58]). These thermistors were calibrated against a platinum-resistance thermometer purchased from Hart Scientific. This thermometer was supplied with a calibration, accurate to $\pm 7$ mK, on the ITS90 temperature scale. The average temperature readings were used to obtain $\Delta = T_b - T_t$ and $T_m = (T_t + T_b)/2$. Small corrections for the temperature gradients in the copper plates were applied. The center temperature $T_c$ was taken to be the average of the temperatures measured with eight thermistors attached to the outside of the side wall at the horizontal mid-plane, equally spaced in the azimuthal direction (see, for instance, Ref. [22] or [59]).

In order to obtain an estimate of the OB values of the Nusselt numbers, a power law $N_{uOB} = N_0 R^{\gamma_{eff}}$ was fitted to the Nusselt-number measurements at relatively small $\Delta$ where $\beta_m \Delta \lesssim 0.05$, adjusting $\gamma_{eff}$ and $N_0$. Such fits yielded values of $\gamma_{eff}$ close to 0.30. All the measured values of $N_u$ regardless of $\beta_m \Delta$ were then divided by the power-law value at the measured Rayleigh numbers to give $N_u/N_{uOB}$ at all $\Delta$.

All measurements reported here were made with many values of $\Delta$ at each of a few constant values of $T_m$ and $P$. In both cells we used ethane at elevated pressures as the fluid. The thermophysical properties were calculated from the formulas given in Ref. [47]. For extensive discussions of the uncertainties of these properties we refer to that paper. It is difficult to determine the absolute errors for the Rayleigh and Nusselt numbers that results from property uncertainties, but we expect that an estimate of a few percent is not unreasonable. Since in the present paper we are concerned only with the ratios $N_u/N_{uOB}$, and since all data are taken as a function of $\Delta$ at a given mean temperature and pressure and evaluated at the same $P$ and $T_m$, property errors cancel to a very large extent.

As indicated above, we do not regard the results for $N_u(Ra)$ to be very accurate because of unknown effects due to the relatively large wall conductivity. Nonetheless we show the results for $\Gamma = 1$ at several $T_m$ and $P$ in Fig. 5 on logarithmic scales. Over a wide range of Ra one sees that they are a few percent higher than the results from Refs. [6] and [7], and we attribute this to the influence of the side-wall conductivity on our data. At the largest Ra our results increase more rapidly with Ra, and data at different $T_m$ and $P$ begin to differ from each other. We attribute this phenomenon to NOB effects.
FIG. 5: (a): The Nusselt number Nu as a function of the Rayleigh number Ra for $\Gamma = 1.00$. Solid circles: $P = 51.72$ bars and $T_m = 40^\circ C$ (Pr = 2.58). Open circles: $P = 51.72$ bars and $T_m = 27^\circ C$ (Pr = 2.99). Solid squares: $P = 51.72$ bars and $T_m = 24^\circ C$ (Pr = 2.71). Open squares: $P = 51.72$ bars and $T_m = 31^\circ C$ (Pr = 3.85). Solid diamonds: $P = 53.79$ bars and $T_m = 40^\circ C$ (Pr = 3.79). Open diamonds: $P = 50.00$ bars and $T_m = 40^\circ C$ (Pr = 2.09). Stars: from Ref. [6] for $\Gamma = 0.5$ after correction for sidewall effects. Plusses: from Ref. [7] for $\Gamma = 0.5$.

FIG. 6: (a): The ratio of the measured Nusselt number $N_{u}$ to the estimate $N_{uOB}$ of the Nusselt number under Boussinesq conditions as a function of the applied temperature difference $\Delta$. (b) The deviation of the center temperature $T_c$ from the mean temperature $T_m$ as a function of $\Delta$. All measurements were made at $T_m = 40.00^\circ C$ and a pressure of 51.72 bar ($P/P_c = 1.062$) where the Prandtl number is 2.58. Open symbols: $\Gamma = 0.50$. Solid symbols: data from Ref. [46] with $\Gamma = 1.00$.

FIG. 7: The ratios $X/X_m$ for $P = 51.72$ bar ($P/P_c = 1.062$) of several property values $X$ at temperatures $T - T_m$ to the value of $X$ at $T_m$ (based on Ref. [47]). (a) and (c): $T_m = 27.00^\circ C$. (b) and (d): $T_m = 40.00^\circ C$. (a) and (b): thermal conductivity $\Lambda$ (short dashed line), density $\rho$ (dotted line), and dynamic viscosity $\eta$ (dash-dotted line). (c) and (d): thermal expansion coefficient $\beta$ (solid line), $\beta$ (long dashed line), and heat capacity at constant pressure $c_P$ (double-dashed dotted line). The reference values $X_m$ for 27(40)$^\circ C$ are $\Lambda_m = 0.07328(0.04343)$ W m$^{-1}$ K$^{-1}$, $\rho_m = 331.12(123.26)$ kg m$^{-3}$, $\eta_m = 4.030(1.502)10^{-5}$ kg s$^{-1}$ m$^{-1}$, $\beta_m = 0.01649(0.03815)$ K$^{-1}$, $c_P,m = 5434(7452)$ J kg$^{-1}$ K$^{-1}$ and mean Prandtl number $Pr_m = 2.99(2.58)$.

2. Aspect-ratio dependence

In Fig. 6 we compare results obtained at a mean temperature $T_m = 40.00^\circ C$ and pressure $P = 51.72$ bar ($P/P_c = 1.062$, Prandtl number $Pr = 2.58$) in the sample of aspect ratio $\Gamma = 0.50$ (open circles) with previously reported results ([46]) for $\Gamma = 1.00$ (solid circles). One sees that the NOB effect on $Nu$ and on the center temperature $T_c$ is within our resolution independent of $\Gamma$. This shows, as expected, that the NOB effects are confined essentially to the boundary layers. The length of the sample interior, which is nearly isothermal (see, however, Ref. [59]) regardless of its length, does not have a large influence.

3. Dependence on fluid properties

Interesting insight into the influence of various property variations with temperature can be gained by measuring $T_c$ and $Nu$ along an isobar on the two sides of the temperature $T_{\phi}(P)$ at which the critical isochore is reached. In Fig. 7 we show the variation along the isobar $P = 51.72$ bar = 1.062 $P_c$ of the thermal conductivity $\Lambda$, etc.
density \( \rho \), dynamic viscosity \( \eta \), thermal expansion coefficient \( \beta \), and heat capacity \( c_p \) for the cases \( T_m = 27.00^\circ C \) (left panels) and 40.00\(^\circ C \) (right panels). These two values are on opposite sides of but not equidistant from, the temperature \( T_\phi = 34.97^\circ C \) where the critical isochore is reached on this isobar.

In Fig. 8 experimental results are presented for \( \Gamma = 1.00 \) at a pressure \( P = 51.72 \) bar (\( P/P_\ast = 1.062 \)). Open symbols: \( T_m = 27.00^\circ C \) (\( Pr = 2.99 \)). Solid symbols: \( T_m = 40.00^\circ C \) (\( Pr = 2.58 \)). These two temperatures are on opposite sides of, but not equidistant from, the temperature \( T_\phi = 34.97^\circ C \) where the critical isochore is reached on this isobar.

FIG. 8: (a): The ratio of the measured Nusselt number \( Nu \) to the estimate \( Nu_{\text{NOB}} \) of the Nusselt number under Boussinesq conditions as a function of the applied temperature difference \( \Delta \). (b) The deviation of the center temperature \( T_c \) from the mean temperature \( T_m \). These measurements were made for \( \Gamma = 1.00 \) at a pressure of 51.72 bar (\( P/P_\ast = 1.062 \)). Open symbols: \( T_m = 27.00^\circ C \) (\( Pr = 2.99 \)). Solid symbols: \( T_m = 40.00^\circ C \) (\( Pr = 2.58 \)). These two temperatures are on opposite sides of, but not equidistant from, the temperature \( T_\phi = 34.97^\circ C \) where the critical isochore is reached on this isobar.

The ratio \( X/X_m \) for \( P = 55.17 \) bar (\( P/P_\ast = 1.132 \)) of several property values \( X \) at temperatures \( T - T_m \) to the value of \( X \) at \( T_m \) (based on Ref. [47]). (a) and (c): \( T_m = 35.00^\circ C \). (b) and (d): \( T_m = 41.00^\circ C \). (a) and (b): thermal conductivity \( \Lambda \) (dashed line), density \( \rho \) (dotted line), and dynamic viscosity \( \eta \) (dash-dotted line). (c) and (d): thermal expansion coefficient \( \beta \) (solid line), \( \beta \) (long dashed line), and heat capacity at constant pressure \( c_p \) (double-dashed dotted line). The reference values \( X_m \) for 35.41\(^\circ C \) are \( \Lambda_m = 0.06674(0.05163) \) W m\(^{-1}\) K\(^{-1}\), \( \rho_m = 282.48(153.43) \) kg m\(^{-3}\), \( \eta_m = 3.168(1.726) \times 10^{-5} \) kg s\(^{-1}\) m\(^{-1}\), \( \beta_m = 0.04177(0.06876) \) K\(^{-1}\), \( c_{p,m} = 9617(12534) \) J kg\(^{-1}\) K\(^{-1}\), and mean Prandtl number \( Pr_m = 4.56(4.20) \).

In Fig. 8 experimental results are presented for \( \Gamma = 1.00 \) at a pressure \( P = 51.72 \) bar (\( P/P_\ast = 1.062 \)). They are for the two mean temperatures \( T_m = 27.00 \) (open circles) and 40.00\(^\circ C \) (solid circles) of Fig. 7 where the Prandtl numbers are 2.99 and 2.58 respectively. In both cases we used \( \Delta \) values sufficiently small so that \( T_{b_1} (T_{b_2}) \) only reaches down (up) to \( T_\phi \) so that the applied temperature difference does not straddle \( T_\phi \). One sees that the NOB effects increase \( Nu \) on both sides of the critical isochore. On the high-temperature side (solid circles) the NOB effect is larger for the same \( \Delta \). This is consistent with the larger variation of the fluid properties at equal values of \( T - T_m \) revealed above in Fig. 7.

The NOB effect on \( T_c \) is of opposite sign on the two sides of the critical isochore. For \( T < T_\phi \) (open circles) NOB conditions increase \( T_c \) above \( T_m \), whereas for \( T > T_\phi \) (solid circles) \( T_c \) is reduced below \( T_m \). This observation, in conjunction with the properties shown in Fig. 7, suggests that for these fluids the temperature drops \( \Delta_{t,b} \) across the boundary layers are determined primarily by \( \beta \) and/or \( c_p \), with \( \Delta_t > \Delta_b \) (\( \Delta_t < \Delta_b \)) when \( \beta \) and/or \( c_p \) are smaller (larger) at the cold top end of the sample then at the warm bottom end. As pointed out before, [43] for the Nusselt number the contributions to the thermal resistance at the two boundary layers add, and it does not matter much whether the larger or smaller contribution comes from one end or the other. Thus, for \( Nu \) the NOB effect is in the same direction in both cases. As was the case for \( Nu \), the NOB effect revealed by \( T_c \) is larger in magnitude above \( T_\phi \) than it is below. Again we attribute this difference primarily to the difference in the variations of the properties shown in Fig. 7.
where the critical isochore is reached on this isobar. The cases $T(\Gamma = 0)$, conditions of Fig. 9 are presented for $\Gamma = 0$. These two temperatures are on opposite sides of and nearly equi-distant from the critical isochore, panels. These two temperatures are also on opposite sides of and nearly equi-distant from the temperature $T\phi = 38.06^\circ C$ where the critical isochore is reached on this isobar.

In Fig. 9 we show the variation along the isobar $P = 55.17$ bar $(P/P_c = 1.132)$ of the various properties for the cases $T_m = 35.00^\circ C$ (left panels) and $41.00^\circ C$ (right panels). These two temperatures are also on opposite sides of and nearly equi-distant from the critical isochore, for this pressure at $T\phi = 38.06^\circ C$. Again the variation of the expansion coefficient and the heat capacity (lower two panels) is much larger than that of the other properties. At a given $|T - T_m|$, all the variations are more similar in magnitude on the two sides of $T\phi$ than they were for the case of Fig. 7.

In Fig. 10 experimental results corresponding to the conditions of Fig. 9 are presented for $\Gamma = 0.50$. They are for the two mean temperatures $T_m = 35.00$ (open circles) and $41.00^\circ C$ (solid circles) where the Prandtl numbers are 4.56 and 4.20 respectively. In both cases we used $\Delta \leq 6$ K so that $T_\ell (T_h)$ reaches down (up) to $T\phi$ when $T_m = 41^\circ C (35^\circ C)$ while the applied temperature difference does not straddle $T\phi$. For this case one sees that the NOB effects on $Nu$ are similar on the two sides of the critical isochore. Again, the NOB effect on $T_\ell$, although of about the same magnitude, is of opposite sign on the two sides. For $T < T\phi$ (open circles, more liquid-like) NOB conditions increase $T_\ell$ above $T_m$, whereas for $T > T\phi$ (solid circles, more gas-like) $T_\ell$ is reduced below $T_m$.

**IV. COMPARISON BETWEEN BOUNDARY-LAYER THEORY AND THE EXPERIMENTAL RESULTS**

We now compare the experimental measurements from the previous section with the boundary layer results presented in section II. In particular, since the comparison with experiments at $T_m = 40^\circ C$ (more gas-like ethane) was already discussed in reference [46], finding good agreement between experiment and the extended BL theory, we devote special attention to the new measurements at $T_m = 27^\circ C$ and $P/P_c = 1.062$ (more liquid-like ethane).

As shown in figure 11, the curve for more liquid-like ethane obtained from BL theory considerably deviates from the experimental data. This is remarkably different from the comparison of BL theory with gaseous ethane, presented in [46], where instead a good agreement was observed. This suggests that even though BL theory reasonably captures NOB effects associated with (OB.1) and (OB.2), further corrections are essential in the present liquid-like ethane case. Among them, deviations from (OB.3) seem to be the natural candidate for the failure of BL theory, since the buoyancy force is not included in the BL equations but apparently affects the thermal convection. Thus, in order to reveal the importance of nonlinear buoyancy in thermal convection, we shall perform direct numerical simulations (DNS) of the RB problem.

**V. DIRECT NUMERICAL SIMULATIONS**

As shown in refs. [45, 60], two-dimensional direct numerical simulations may be useful for the study of the tiny NOB effects which occur in RB convection in liquids. In particular, even with the restrictions to two-dimensional geometry and to incompressibility of the fluid flow, the effects on the center temperature and the Nusselt number shift could be reasonably captured in the cases of water and glycerol. However, the liquid-like ethane just above the critical pressure has a stronger temperature dependence of the density than water and glycerol. To quantify this, a comparison between the ethane properties around $T_m = 27^\circ C$ and $P/P_c = 1.062$ with water and glycerol around $T_m = 40^\circ C$ is reported in figure 12. For the case of ethane the incompressible flow approximation seems to be questionable, or at least less justified. But we will show that adopting the same approach used for water and glycerol [45, 60] also proves to be useful to study NOB effects in ethane and the results are well consistent with experiment.

Further insight into the problem is given by considering several cases of artificial ethane-like fluids, namely fluids which have only one, or some, of their material properties dependent on temperature, while the others are kept constant. In particular, as discussed in the previous section, we will examine the relevance of the nonlinear temperature dependence of buoyancy on the center tem-
temperature shift $T_c - T_m$ and take full notice of violating OB.3, which in contrast assumes constant $\partial \rho / \partial T$. We remind that this cannot be taken into account in the extended BL theory presented in section II, while DNS can well include it.

A. Numerical simulation approach

To handle the numerical effort we restrict ourselves to incompressible and even two dimensional flow. The equations governing non-Oberbeck-Boussinesq convection in incompressible fluid flow are the incompressibility condition

$$ \partial_t u_i = 0, \quad (14) $$

the Navier-Stokes equation

$$ \rho_m (\partial_t u_i + u_j \partial_j u_i) = -\partial_i p + \partial_j (\eta (\partial_j u_i + \partial_i u_j)) + g \rho_m (1 - \rho/\rho_m) \delta_{i3}, \quad (15) $$

and the heat-transfer equation

$$ \rho_m c_p,m (\partial_t T + u_j \partial_j T) = \partial_j (\Lambda \partial_j T). \quad (16) $$

Here, $\delta_{i3}$ is the Kronecker symbol. The density is assumed to be constant and its value $\rho_m$ is fixed at that of the temperature $T_m$, except in the buoyancy term, where the full nonlinear temperature dependence of $\rho(T)$ is implemented. The dynamic viscosity $\eta(T)$ and the heat conductivity $\Lambda(T)$ are also both temperature and thus space dependent. The isobaric specific heat capacity $c_p$ is assumed to be constant, its value being $c_{p,m}$.
(in contrast to real ethane). The experimentally known
temperature dependences of $\rho$, $\eta$, $\Lambda$ and the values of the
parameters $\rho_m$, $c_{p,m}$ for ethane are given in [47] and, for
better reference, are reported in Table I in the specific
form implemented in our DNS.

For consistency with the experimental measurements
and with the BL theoretical analyses presented above for
liquid-like ethane, we chose the arithmetic mean temper-
ature shift $\hat{\beta}(T_m)\Delta T$ instead, see Eq.(15) and T ab. I. Fi-
ter the various material properties on the center
temperature shift $T_c - T_m$ as a function of $\Delta$ quite reasonably.
The quality of the agreement with the available ethane
data is similar to the one we have observed for glycerol
[45] and for water [60]. This also serves as a further val-
ification of our numerical approach.

We note that for water and for glycerol the $T_c$ shift ob-
tained by the extended BL theory [43] is nearly the same
as calculated by DNS (see [45, 60]). In contrast, for the
liquid-like ethane, the extended and even compressible
BL theory only provides the right trend in the shift, but
can not capture its amplitude (see again Fig. 11). This
observation supports our guess on the relevance of the
nonlinear $T$-dependence of $\rho(T)$ and thus of buoyancy.
This latter is fully included in the DNS, as described in
the previous subsection, while in BL theory it cannot be
taken care of.

One of the advantages of the DNS as compared to real
experiments is that the material properties are easily and
independently tunable. Therefore, the dynamics of hypo-
thetical ethane-like liquids can also be addressed. In the
next subsection we shall see how this approach is useful
in understanding the effects of the temperature depen-
dence of the various material properties on the center
temperature shift.

C. $T_c$ shift in hypothetical fluids

To obtain more insight into the physical origin of the
non-Oberbeck-Boussinesq $T_c$-shift, we consider NOB cor-
corrections for hypothetical ethane-like fluids in which at
least one of the temperature dependences of $\kappa(T)$, $\nu(T)$,
$\hat{\beta}(T)$ is switched off, and fixed at the OB values $\kappa_m$, $\nu_m$, $\beta_m$. The quantity $\hat{\beta}(T)$, defined in eq. (2), is useful
for the classification of the hypothetical fluids discussed
in the following sections, but $\hat{\beta}(T)$ is not explicitly in-
troduced into the DNS, in which the density difference
$\rho(T) - \rho_m$ is taken instead, see Eq.(15) and Tab. I. Fi-
ally, we remind that $\hat{\beta}(T) = \beta_m$, the usual thermal expansion coefficient, if the fluid density is a linear func-
tion of the temperature around $T_m$ (i.e., if the conditions

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{The normalized center temperature shift $(T_c - T_m)/\Delta$ versus the temperature difference $\Delta$ for several hypothetical liquids. We consider the following six hypothetical liquids: $(\hat{\beta}(T), \kappa(T), \nu(T))$, $(\beta_m, \kappa(T), \nu(T))$, $\hat{\beta}(T, \kappa(T), \nu_m)$, $\beta_m, \kappa(T), \nu_m)$, $\beta(T, \kappa_m, \nu(T))$ and $(\beta_m, \kappa_m, \nu(T))$. In particular in each panel we compare two
cases, which differ only by their buoyancy’s $T$-dependence, i.e., $\beta(T)$ instead of $\beta_m$. The symbols indicate the simu-
lation results at Rayleigh number $Ra = 10^6$. The circles (○) correspond to the cases in which the full temperature dependence of the buoyancy $(T - T_m) \hat{\beta}(T)$ is taken into account, while the triangles ($\triangle$) represent the cases where only the linear temperature dependence $(T - T_m) \beta_m$ is
considered. The solid line shows the prediction of boundary layer theory with an incompressible flow assumption [43]. The dashed line stems from the solution with no convective flow $\partial_z (\nu \partial_z T) = 0$ with boundary conditions $T = T_i$ at $z = 0$ and $T = T_i$ at $z = L$. See also Table II.}
\end{figure}
The pressure normalized by its critical value is $P/P_c = 1.062$. The effective kinematic viscosity, the effective thermal diffusivity, and the buoyancy are written in a polynomial form as $\nu(T) = \eta(T)/\rho_m = \sum_{n=0}^{m} A_n (T - T_m)^n$ [m$^2$/s], $\kappa(T) = \Lambda(T)/(\rho_m c_p, m) = \sum_{n=0}^{m} B_n (T - T_m)^n$ [m$^2$/s$^2$], and $g(1 - \rho(T)/\rho_m) = \sum_{n=1}^{m} C_n (T - T_m)^n$ [m/s$^2$], respectively. Using the leading coefficient for the buoyancy force, we can write the Rayleigh number as $Ra = \beta_m L^2 \Delta/(\nu_m \kappa_m)$, where $\beta_m = C_1$, $\nu_m = A_0$ and $\kappa_m = B_0$, which coincides with the usual OB definition. The polynomial expansions for $\beta(T)$ and $\beta(T)$ are $g\beta(T) = \sum_{n=1}^{m} C_n (T - T_m)^n$ and $g\beta(T) = (\rho_m/\rho(T)) \cdot \sum_{n=1}^{m} nC_n (T - T_m)^n$.

In DNS we have restricted to incompressible flow. Instead, $\beta$-fluids and extended BL theory is used to correctly describe the NOB effect once the assumption (OB.3) holds. In that case the thermal expansion coefficient $\beta(T) = \beta_m/[1 - (T - T_m)\beta_m]$ still depends on $T$ unless $\beta \Delta \ll 1$.

We now discuss our main findings. The change in $T_c$ for the hypothetical liquids with $\beta_m$ relative to the ones with $\beta(T)$ is comparable for given $\Delta$. More importantly, we find that the deviation $(T_c - T_m)/\Delta$ calculated with DNS is well captured by the BL theory for the cases of artificial $\beta_m$-fluids, i.e., for fluids where $\rho(T)$ is assumed to be a linear function of $T$. In contrast, BL theory is always far from the $T_c$ shift obtained for the more real $\beta(T)$-ethane. This indicates that the extended BL theory well captures the NOB effect once the assumption (OB.3) is satisfied, even if (OB.1) and (OB.2) are violated, but it does not correctly describe the NOB effects, if (OB.3) is violated. Furthermore, the DNS results reveal that the $T_c$-shift is always enhanced if a temperature dependence of the thermal expansion function $\beta(T)$ determines the buoyancy, i.e., if $\rho(T)$ depends nonlinearly on $T$.

2. Mirror transformation $\hat{\beta}(T) \to \hat{\beta}(2T_m - T)$

To quantitatively appreciate the effect of the temperature dependence in each material property individually also at different Rayleigh numbers, $Ra = 10^6$-$10^8$, we list the $T_c$ shifts for several hypothetical ethane-like fluids in Tab. II. Since here our primary concern is the influence of the thermal expansion function, besides the $\beta(T)$- and $\beta_m$-fluids a new class of hypothetical fluids is introduced. We consider $\beta(2T_m - T)$-fluids obtained by the mirror transformation $\hat{\beta}(T) \to \hat{\beta}(2T_m - T)$. As schematically shown in figure 14, this transformation reverts the nonlinearity in the buoyancy force with respect to $T - T_m$. The comparison between the cases $(\hat{\beta}(T), \kappa_m, \nu_m)$ and $(\hat{\beta}(2T_m - T), \kappa_m, \nu_m)$ - cases 4 and 8 in table II - shows that the effect of the mirror transformation on the out-

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</table>

**Table I**: Expansion coefficients of material properties of ethane around the temperature $T_m = 27^\circ$C adapted from [47].
put parameter $T_c$ is to change the sign of $(T_c - T_m)/\Delta$ while preserving its modulus. Furthermore, the deviation of $(T_c - T_m)/\Delta$ for all $\beta_m$-fluids (cases 9, 10, 11) relative to the $\beta(T)$-fluids (cases 1, 2, 3) is always positive, while it is always negative relative to the $\beta(2T_m-T)$-fluids (cases 5, 6, 7). These features hold at all the studied $Ra$ numbers. Therefore, we conclude that the shift $T_c - T_m$ is sensitive to the sign of the slope of $\beta(T)$ or, equivalently, to the sign of the nonlinear term in the buoyancy factor $1 - \rho(T)/\rho_m$. More precisely speaking, the mirror transformation changes the signs of the even order coefficients $C_2$ and $C_4$ defined in Tab. I. Obviously $C_2$ is the larger and thus the most relevant coefficient. These features are absent in water and in glycerol, because for those the temperature dependence of $\rho(T)$ is much less pronounced.

Looking at Tab. II in more detail we find that all changes in $(T_c - T_m)/\Delta$, which stem from the nonlinear $T$-dependence of the buoyancy force, i.e., from the differences between $\beta_m$ and $\hat{\beta}(T)$ (or $\beta(2T_m-T)$ but having the same temperature dependences of $\kappa$ and $\nu$, are comparable. To emphasize this feature, we look at the differences $T_c|\beta(T) - T_c|\beta_m$ (or $-(T_c|\beta(2T_m-T) - T_c|\beta_m)$) and plot them versus $\Delta$ as well as versus $Ra$, see figure 15. We find a good collapse of the data onto a single curve for various temperature dependences of $\kappa$ and $\nu$. In particular, the comparison between the middle and bottom panels of Fig. 15 leads to the relation $T_c|\beta(T) - T_c|\beta_m = -(T_c|\beta(2T_m-T) - T_c|\beta_m)$, which indicates that the $T_c$ change is dominated by the quadratic term in $1 - \rho(T)/\rho_m$, but is almost independent of $\kappa$ and $\nu$. This observation may be important for further attempts to improve the extended NOB BL theory.

VI. CONCLUSIONS

In this paper we have first presented in full detail the extension of boundary-layer theory to the case of compressible NOB fluids in a Rayleigh-Bénard system. The theory predicts a deviation of the center temperature $T_c$ from the arithmetic mean temperature $T_m$ between the top and bottom plates, i.e., $T_c - T_m \neq 0$.

Second, the theory has been tested against new experimental data for ethane near the critical point in its more liquid-like phase. Data come from a series of experiments.
FIG. 15: Effect of the temperature dependence of the thermal expansion coefficient \( \beta \) on the shift of the center temperature for several hypothetical fluids. DNS results for the normalized temperature difference (\( T_c|\beta(T) - T_c|\beta_m / \Delta \)) with the same temperature dependence of the thermal diffusivity \( \kappa \) and the kinematic viscosity \( \nu \) are plotted; here \( T_c|\beta_m \) denotes the center temperature with the full temperature dependence of the buoyancy \( g(1 - \rho(T)/\rho_m) \) as given in Table I, and \( T_c|\beta_m \) denotes that with the linear temperature dependence \( g(1 - \rho(T)/\rho_m) = C_1(T - T_m) \) only. The top panel shows \( (T_c|\beta(T) - T_c|\beta_m) / \Delta \) versus \( \Delta \) at fixed Rayleigh number \( Ra = 10^6 \). The solid line shows the linear fit \( 1.473 \cdot 10^{-3} \text{K}^{-1} \times \Delta \) for the case \( \beta(T), \kappa_m, \nu_m \) (see Table II). The middle panel shows \( (T_c|\beta(T) - T_c|\beta_m) / \Delta \) versus \( Ra \) at fixed temperature difference \( \Delta = 10 \text{K} \). The bottom panel has the same parameters as the middle one, except showing \( -(T_c|\beta(2T_m - T) - T_c|\beta_m) / \Delta \) for the case of mirror transformed \( \beta \).

in cylindrical cells of aspect ratio \( \Gamma = 0.5 \) and 1, reaching \( Ra \) numbers \( O(10^{10}) \). The experimental measurements at \( T_m = 27^\circ \text{C}, P/P_c = 1.062 \) and \( \Gamma = 1 \) have been chosen for comparison with those at \( T_m = 40^\circ \text{C} \). Contrary to the good agreement observed for the case of gas-like ethane [43], the BL theory here gives much smaller values of the center temperature shift as experiment.

Third, direct numerical simulations DNS, based on \( T \)-dependent material parameters but still within the incompressible approximation and a two-dimensional domain have been performed to get more insight into the observed discrepancy between experiment and extended BL theory. The DNS results provide a satisfactory agreement with experiment both in the gas-like as well as in the liquid-like cases. Several hypothetical ethane-like fluids have been investigated too. Our analysis shows that the extended BL predictions fail whenever the non-linear temperature dependence of the density \( \rho(T) \) is implemented in the numerical simulations. Furthermore, if the dependence of \( \beta(T) \) on \( T \) dominates the NOB effects, the sign of the linear term in the effective expansion function \( \beta(T) \) is responsible for the sign of the variations of \( T_c \) as compared to the OB value \( T_c = T_m \). But even if the difference \( \beta(T) - \beta_m \) is a non-linear function of the temperature, the tiny \( T_c \) shift detected in our simulations is proportional to \( T_c|\beta(T) - T_c|\beta_m \).

Our results can be summarized in the following physical picture of the surprising phenomenon that the center temperature \( T_c \) in liquid-like ethane near its critical point increases, the center becomes warmer than the arithmetic mean temperature \( T_m \), while in gas-like ethane the center gets colder, \( T_c \) is less than \( T_m \). Namely, in the liquid-like case the buoyancy \( \propto (T - T_m)\beta(T) \) is larger at the bottom and smaller at the top, supporting the uprising warmer plumes more than the down coming colder top plumes. This brings predominantly hotter material into the bulk. For gas-like ethane the buoyancy is larger at the cooler top, which accelerates the downgoing cold plumes with more preference than the uprising warm plumes from the bottom, that experience a weaker buoyancy. This in turn brings more cooler material into the bulk, leading to \( T_c < T_m \). It is the sign of the slope of \( \beta \) (or of \( \beta \)), which is the relevant quantity.

The more general lesson which can be drawn from this paper is that there is a plethora of origins of NOB corrections. Which one dominates can only be said by having a detailed look at the temperature dependence of all material parameters. Both the extended BL theory and 2D DNS are useful tools to judge which temperature dependence is the most relevant one or whether they all matter, as we have now often seen.

VII. ACKNOWLEDGMENTS

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APPENDIX A: BOUNDARY-LAYER EQUATIONS

1. Viscous boundary-layer

Consider two-dimensional flow over a flat plate, such that the main wind velocity \( U \) does not depend on \( x \) and \( \lim_{z \to \infty} u_x(x,z) = U \). Then, the \( x \)-momentum near the wall is governed by [50]

\[
\rho \left\{ u_x \frac{\partial u_x}{\partial x} + u_z \frac{\partial u_x}{\partial z} \right\} = \eta \left\{ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2} \right\} + \frac{\partial \eta}{\partial z} \left\{ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right\} + \frac{\partial}{\partial x} \left\{ \eta \left[ \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right] \right\},
\]

where \( \eta \) is the dynamic shear viscosity and \( \zeta \) the volume expansion viscosity.

To appraise the dominant structure of equation (A1) at sufficiently large Reynolds numbers, we follow Prandtl’s scaling:

\[
\begin{align*}
x &= L \hat{x}, \\
z &= \frac{L}{\sqrt{Re}} \hat{z}, \\
u_x &= U \hat{u}_x, \\
u_z &= \frac{U}{\sqrt{Re}} \hat{u}_z,
\end{align*}
\]

where \( L \) denotes the typical length of the plate in flow direction and \( Re = LU \rho_m/\eta_m \) the Reynolds number. The index \( m \) indicates that the fluid properties are evaluated at a thermodynamic reference state \( (T_m, P_m, \rho_m) \), which is adopted in the nondimensionalization of \( \rho \equiv \rho_m \hat{\rho} \) and \( \eta \equiv \eta_m \hat{\eta} \). Then, substituting (A2)-(A5) into (A1), one finds

\[
\begin{align*}
\hat{\rho} \left\{ \hat{u}_x \frac{\partial \hat{u}_x}{\partial \hat{x}} + \hat{u}_z \frac{\partial \hat{u}_x}{\partial \hat{z}} \right\} &= \hat{\eta} \left\{ \frac{1}{Re} \frac{\partial^2 \hat{u}_x}{\partial \hat{x}^2} + \frac{1}{Re} \frac{\partial^2 \hat{u}_x}{\partial \hat{z}^2} \right\} \\
+ \hat{\eta} \frac{\partial}{\partial \hat{z}} \left\{ \frac{\partial \hat{u}_x}{\partial \hat{z}} + \frac{\partial \hat{u}_z}{\partial \hat{x}} \right\} \\
+ \frac{1}{Re} \frac{\partial}{\partial \hat{x}} \left\{ \hat{\eta}_0 \left[ \frac{\partial \hat{u}_x}{\partial \hat{x}} + \frac{\partial \hat{u}_z}{\partial \hat{z}} \right] \right\},
\end{align*}
\]

where \( \hat{\eta}_0 = \hat{\eta} \left[ \frac{1}{Re} \left( \frac{\partial \hat{u}_x}{\partial \hat{x}} + \frac{\partial \hat{u}_z}{\partial \hat{z}} \right) \right] \). Clearly, all terms on the left-hand side of equation (A6) are of order unity. However, this is not the case on the right-hand side of (A6): only the viscous contributions involving \textit{transverse} gradients of \( u_x \) are of order 1; the remaining terms are of order \( 1/Re \).

Therefore, at large \( Re \), the dominant part of the \( x \)-momentum equation is given by

\[
\begin{align*}
\hat{\rho} \left\{ \hat{u}_x \frac{\partial \hat{u}_x}{\partial \hat{x}} + \hat{u}_z \frac{\partial \hat{u}_x}{\partial \hat{z}} \right\} &= \hat{\eta} \frac{\partial^2 \hat{u}_x}{\partial \hat{z}^2} + \frac{\partial \hat{\eta}}{\partial \hat{z}} \frac{\partial \hat{u}_x}{\partial \hat{z}}.
\end{align*}
\]

2. Thermal boundary-layer

Consider again two-dimensional, subsonic, and steady flow over a flat plate. In the framework of boundary-layer theory, energy conservation leads to the following equation for the entropy per mass [41]:

\[
\begin{align*}
\frac{u_x \rho T}{\partial x} + u_z \rho T \frac{\partial s}{\partial z} &= \frac{\partial}{\partial z} \left\{ \Lambda \frac{\partial T}{\partial z} \right\}.
\end{align*}
\]

Letting \( s = s(T, \rho) \), the entropy gradient (say, its \( \partial_z s \) component) can be expressed as

\[
\frac{\partial s}{\partial z} = \left( \frac{\partial s}{\partial T} \right)_\rho \frac{\partial T}{\partial z} + \left( \frac{\partial s}{\partial \rho} \right)_T \frac{\partial \rho}{\partial z}.
\]

The first contribution is directly associated with the isochoric specific heat (per mass) of the gas

\[
T \left( \frac{\partial s}{\partial T} \right)_\rho \equiv cv \equiv \frac{c_p}{\gamma}.
\]

The second contribution follows from a Maxwell relation,

\[
\left( \frac{\partial s}{\partial \rho} \right)_T = -\frac{1}{\rho^2} \left( \frac{\partial P}{\partial T} \right)_\rho = \frac{1}{\rho^2} \frac{c_p}{\gamma} \frac{\gamma - 1}{\beta}.
\]

Thus, the left-hand side of equation (A7) can be written as

\[
u_x \rho T \frac{\partial s}{\partial x} + u_z \rho T \frac{\partial s}{\partial z} = \rho \frac{c_p}{\gamma} \frac{\gamma - 1}{\beta} \left( u_x \frac{\partial T}{\partial x} + u_z \frac{\partial T}{\partial z} \right) - \frac{c_p}{\gamma} \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial z}.
\]

Finally, using the continuity equation \( u_x \partial_x \rho + u_z \partial_z \rho = -\rho \left( \partial_z u_x + \partial_x u_z \right) \), one finds

\[
u_x \frac{\partial T}{\partial x} + u_z \frac{\partial T}{\partial z} + \frac{\gamma - 1}{\beta} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) = \frac{\gamma}{\rho c_p} \frac{\partial \rho}{\partial z} \left\{ \Lambda \frac{\partial T}{\partial z} \right\}.
\]

Note that the limiting case of liquids (namely \( \gamma = 1 \)) is fully accounted by this equation.

\footnote{Note that the term involving \( \nabla \cdot \tilde{u} = \frac{\partial \tilde{u}_x}{\partial x} + \frac{\partial \tilde{u}_z}{\partial z} \) is of order \( 1/Re \), as long as \( \zeta \) and \( \eta \) are of the same order of magnitude. Indeed, \( \zeta/\eta \) is of order unity if acoustical effects (\( Ma \ll 1 \)) and chemical reactions do not take place in the fluid. For situations in which \( \zeta > \eta \), see [41,Section 81].}
APPENDIX B: SELF-SIMILARITY ANSATZ

1. Viscous boundary-layer

In the stream-function representation (8), the longitudinal velocity \(u_x\) is expressed as

\[
u_m \frac{\partial}{\partial z} \left( \ell_c U_c \frac{\hat{\Psi}}{\rho} \right) = U_c \frac{\hat{\Psi}'}{\rho}
\]

and its spatial derivatives are

\[
\frac{\partial u_x}{\partial x} = -\frac{\nu_m}{2\ell_c^2} \frac{\dot{Z}}{\rho^2} \left( \hat{\Psi}'' - \hat{\Psi}' \right), \quad (B1)\\
\frac{\partial u_x}{\partial z} = \frac{U_c}{\ell_c} \frac{1}{\rho^2} \left( \hat{\Psi}'' - \hat{\Psi}' \right), \quad (B2)\\
\frac{\partial^2 u_x}{\partial z^2} = \frac{U_c}{\ell_c} \frac{1}{\rho^2} \left( \hat{\Psi}'''' - 2\hat{\Psi}'' \right) + \left[ \frac{2(\hat{\rho}')^2 - \hat{\rho}'''}{\hat{\rho}'} \right] \hat{\Psi}'. \quad (B3)
\]

Likewise, from equation (9), the transverse velocity \(u_z\) reads

\[
u_m \frac{1}{2\ell_c^2} \frac{\dot{Z}}{\rho^2} \left( \hat{\Psi}'' - \hat{\Psi}' \right) + \hat{\rho}' \hat{\Psi}'', \quad (B4)
\]

Thus, the advective contributions in Prandtl’s equation (5) can be written as

\[
\rho \left\{ u_x \frac{\partial u_x}{\partial x} + u_z \frac{\partial u_z}{\partial z} \right\} = -\frac{1}{2} \frac{\nu_m}{\ell_c} \left\{ \hat{\Psi}'' - \hat{\Psi}' \right\} + \left[ \frac{2(\hat{\rho}')^2 - \hat{\rho}'''}{\hat{\rho}'} \right] \hat{\Psi}'. \quad (B5)
\]

Now consider the viscous contributions:

\[
\frac{\partial}{\partial z} \left\{ \frac{\eta}{\ell_c} \frac{\partial u_x}{\partial z} \right\} = \eta \frac{\partial^2 u_x}{\partial z^2} + \left[ \frac{\partial \eta}{\partial \eta} \right] \frac{\partial u_x}{\partial z}. \quad (B6)
\]

Since

\[
\frac{\partial \eta}{\partial z} = \frac{\eta_m}{\ell_c} \left\{ \left( \frac{\partial \hat{\eta}}{\partial \hat{\Theta}} \right) \hat{\Theta}' + \left( \frac{\partial \hat{\eta}}{\partial \hat{\rho}} \right) \hat{\rho}' \right\}, \quad (B7)
\]

insertion of (B2) and (B3) into (B6) leads to

\[
\frac{\hat{\rho} \ell_c^2}{\eta_m U_c} \frac{\partial}{\partial z} \left\{ \eta \frac{\partial u_x}{\partial z} \right\} = \hat{\eta} \hat{\Psi}'' + \left[ \eta' - 2\hat{\rho}' \hat{\eta} \right] \hat{\Psi}'' + \left[ \frac{2(\hat{\rho}')^2 - \hat{\rho}'''}{\hat{\rho}'} \right] \hat{\eta} \hat{\Psi}'. \quad (B8)
\]

Therefore, by substituting (B5) and (B8) into (5) one finally obtains equation (11).

2. Thermal boundary-layer

In the same spirit as above, inserting (B1), (B4), and

\[
\frac{\partial}{\partial z} \left\{ \Lambda \frac{\partial T}{\partial z} \right\} = \frac{\Lambda_m}{\ell_c} \left\{ \hat{\Lambda} \hat{\Theta}'' + \hat{\Lambda}' \hat{\Theta}' \right\}
\]

into equation (6) one finds

\[
\frac{\nu_m}{2\ell_c^2} \frac{\dot{\Theta}'}{\rho} \hat{\Theta}' + \frac{\nu_m}{2\ell_c^2} \frac{\gamma - 1}{\beta} \frac{\hat{\rho}'}{\rho^2} \hat{\Theta} = \frac{\gamma}{\rho} \frac{1}{\Pr} \left\{ \hat{\Lambda} \hat{\Theta}'' + \hat{\Lambda}' \hat{\Theta}' \right\}, \quad (B9)
\]

where \(\hat{\beta} = \beta \Delta\) and \(\Pr = (\nu_m \rho_m c_p m)/\Lambda_m\). Thus

\[
\hat{\Lambda} \hat{\Theta}'' + \left\{ \frac{1}{2} \frac{\hat{c}_p}{\gamma} \frac{\Pr}{\Lambda} \hat{\Psi} + \hat{\Lambda}' \right\} \hat{\Theta}' - \frac{\gamma - 1}{2\gamma} \frac{\hat{c}_p}{\beta} \frac{\hat{\rho}'}{\rho} \Pr \hat{\Psi} = 0.
\]

Here, substituting \(\hat{\rho}'\) by (12) one finally obtains

\[
\hat{\Lambda} \hat{\Theta}'' + \left\{ \frac{1}{2} \frac{\hat{c}_p}{\gamma} \frac{\Pr}{\Lambda} \hat{\Psi} + \hat{\Lambda}' \right\} \hat{\Theta}' = 0.
\]

Note that the limiting case of liquids (namely \(\gamma = 1\)) is fully accounted by this equation.

APPENDIX C: NUMERICAL RESULTS ON NUSSELT NUMBERS IN REAL AND HYPOTHETICAL ETHANE FLUIDS

For completeness, in table III the NOB corrections in the Nusselt number are given, resulting from the numerical simulations of real and hypothetical ethane. The corresponding NOB corrections of the center temperature had already been shown in table II.

TABLE III: Simulation results of the relative deviation of the Nusselt number of $\Delta = 10$K for several hypothetical fluids. At the Rayleigh numbers of case $\beta(T)\kappa(T)\nu(T)$ $100(Nu/Nu_{OB} - 1)$ $100(Nu/Nu_{OB} - 1)$ $100(Nu/Nu_{OB} - 1)$ at $Ra = 10^6$ at $Ra = 10^7$ at $Ra = 10^8$

<table>
<thead>
<tr>
<th>case</th>
<th>$\bar{\beta}$</th>
<th>$\kappa$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
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<td>1 (NOB)</td>
<td>$\bar{\beta}(T)$</td>
<td>$\kappa(T)$</td>
<td>$\nu(T)$</td>
</tr>
<tr>
<td>2</td>
<td>$\bar{\beta}(T)$</td>
<td>$\kappa(T)$</td>
<td>$\nu_n$</td>
</tr>
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<td>$\kappa_m$</td>
<td>$\nu(T)$</td>
</tr>
<tr>
<td>4</td>
<td>$\bar{\beta}(T)$</td>
<td>$\kappa_m$</td>
<td>$\nu_n$</td>
</tr>
<tr>
<td>5</td>
<td>$\bar{\beta}(2T_m - T)$</td>
<td>$\kappa(T)$</td>
<td>$\nu(T)$</td>
</tr>
<tr>
<td>6</td>
<td>$\bar{\beta}(2T_m - T)$</td>
<td>$\kappa(T)$</td>
<td>$\nu_n$</td>
</tr>
<tr>
<td>7</td>
<td>$\bar{\beta}(2T_m - T)$</td>
<td>$\kappa_m$</td>
<td>$\nu(T)$</td>
</tr>
<tr>
<td>8</td>
<td>$\bar{\beta}(2T_m - T)$</td>
<td>$\kappa_m$</td>
<td>$\nu_n$</td>
</tr>
<tr>
<td>9</td>
<td>$\beta_m$</td>
<td>$\kappa(T)$</td>
<td>$\nu(T)$</td>
</tr>
<tr>
<td>10</td>
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</tr>
<tr>
<td>11</td>
<td>$\beta_m$</td>
<td>$\kappa_m$</td>
<td>$\nu(T)$</td>
</tr>
<tr>
<td>12 (OB)</td>
<td>$\beta_m$</td>
<td>$\kappa_m$</td>
<td>$\nu_n$</td>
</tr>
</tbody>
</table>


