RECENT DEVELOPMENTS IN
RAYLEIGH-BÉNARD CONVECTION

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ABSTRACT: This review summarizes results for Rayleigh-Bénard convection which have been obtained over the past decade or so. It concentrates on convection in compressed gases and gas mixtures with Prandtl numbers near one and smaller. In addition to the classical problem of a horizontal stationary fluid layer heated from below, it also covers briefly convection in such a layer with rotation about a vertical axis, with inclination, and with modulation of the vertical acceleration.

CONTENTS

INTRODUCTION ................................................. 2

RECENT DEVELOPMENTS ......................................... 4

THEORETICAL ANALYSIS ........................................ 6
  Ideal straight rolls and their stability .................................. 7
  Order parameter equations .............................................. 9
  Phase-diffusion equation .............................................. 10
  Numerical simulations ............................................... 12

FLUCTUATIONS .................................................. 12

IDEAL STRAIGHT ROLLS .......................................... 15

CIRCULAR CELLS ............................................... 20
  Fun-Arn patterns ................................................... 20
  Targets ......................................................... 22
  Giant spirals ..................................................... 24
1 INTRODUCTION

Fluid motion driven by thermal gradients (thermal convection) is a common and important phenomenon in nature. Convection is a major feature of the dynamics of the oceans, the atmosphere, and the interior of stars and planets (Busso, 1978, 1989; Geiling, 1998). It is also important in numerous industrial processes. For many years, the quest for the understanding of convective flows has motivated numerous experimental and theoretical studies.

In spatially extended systems convection usually occurs when a sufficiently steep temperature gradient is applied across a fluid layer. The spatial variation of a convection structure often is referred to as a pattern. The nature of such convection patterns is at the center of this review. Pattern formation is determined by nonlinear aspects of the system under study. For this reason the elucidation of pattern formation is a challenging problem in condensed-matter physics as well as in fluid mechanics. Pattern formation is common also in many other spatially extended nonlinear nonequilibrium systems in physics, chemistry, and biology (Manneville, 1990; Cross & Hohenberg, 1993). Patterns observed in diverse systems are often strikingly similar, and their understanding in terms of general, unifying concepts has long been a main direction of research (Cross & Hohenberg, 1993; Newell et al., 1993).

Many fundamental aspects of patterns and their instabilities have been studied intensively over the past three decades in the context of Rayleigh-Bénard convection (RBC). In a traditional RBC experiment a horizontal fluid layer of height $d$ is confined between two thermally well conducting, parallel plates. When the difference $\Delta T = T_b - T_t$ between the bottom-plate temperature $T_b$ and the top-plate temperature $T_t$ exceeds a critical value $\Delta T_c$, the conductive motionless state is unstable and convection sets in. The simplest pattern which can occur is that of straight, parallel convection rolls with a horizontal wavelength $\lambda \approx 2d$ (wave number $q \approx \pi/d$). Such rolls can be found near onset; however, as the dimensionless distance $\varepsilon \equiv \Delta T/\Delta T_c - 1$ increases, the patterns often become progressively more complicated, and thus also more interesting.

Rayleigh-Bénard convection is perhaps the most thoroughly investigated and understood pattern-forming system. The experimental set up is simple in principle and the basic physical mechanism (buoyancy vs. dissipation) well understood. For the standard description in terms of the Oberbeck-Boussinesq equations,
Eqs. 1 and 2 below, only two nondimensionalized control parameters are sufficient (Busse, 1978, 1989). The first is the Rayleigh number $R = \alpha g \Delta T d^3 / (\nu \kappa)$ with $\alpha$ the thermal expansion coefficient, $\kappa$ the thermal diffusivity, $\nu$ the kinematic viscosity, and $g$ the acceleration of gravity. Convection starts (under ideal conditions) at the critical value $R_c = 1708$. The second parameter is the Prandtl number $\sigma = \nu / \kappa$, which can be viewed as the ratio of the vertical thermal diffusion time $t_v = d^2 / \kappa$ to the vertical viscous relaxation time $t_v = d^2 / \nu$. It measures the relative importance of the nonlinear terms in the Boussinesq equations, namely those terms describing temperature and momentum advection.

There are several recent reviews of Rayleigh-Bénard convection. In the present one we will focus on new developments during the last decade or so. Let us, however, briefly outline some of the seminal earlier results which were of major importance for the later work. Quite early it was established theoretically that the stable pattern in an infinitely extended layer of a Boussinesq fluid close to onset (see Sect. 3 below) consists of straight, parallel rolls of wavenumber $q$ (Schlüter et al, 1965). Further above onset, the stability regimes of these rolls in the $R - q$ space as functions of $\sigma$ (the "Busse balloon") are well understood owing to the impressive work by Busse and coworkers (Busse, 1978, 1989). The value of $\sigma$ varies widely for different experimental fluids, from $O(10^{-2})$ for liquid metals to values near one for gases and for liquid helium, to the range from 2 to 12 for water, and into the 1000's for silicone oil (see de Bruyn et al, 1996, and references therein). Although $R_c$, the critical wavevector $q_c = 3.117$, and the patterns in the close vicinity of onset ($R \approx R_c$) are independent of $\sigma$, this does not apply to the subsequent bifurcations which occur with increasing $R$. For example one finds an oscillatory secondary instability at medium and small Prandtl numbers, in contrast to a stationary bimodal ("knot") bifurcation at large $\sigma$ (Busse, 1978, 1989). Not too far from threshold the Busse balloon was found to agree well with the experiments for large $\sigma$ (water) (Busse & Whitehead, 1971) and reasonably well for gases with $\sigma \approx 1$ (Croquette, 1989a).

Although ideal periodic patterns can be created in experiments, "natural" convective patterns, particularly when they form in the presence of lateral walls, typically are disordered and develop persistent spatio-temporal dynamics as $\epsilon$ increases. Snapshots of such patterns are characterized by local roll patches with grain boundaries and point defects (dislocations).2 These spatio-temporal chaotic patterns are irregular in time and in the horizontal plane, but they maintain a relatively simple structure in the vertical direction. They should be contrasted with fully developed turbulence (Frisch, 1995), at very large $R$, which is disordered in three spatial dimensions and not the topic of this review.

For the description of nonuniform patterns not too far from threshold various reductions of the original hydrodynamical equations have proven to be useful (Manneville, 1990; Cross & Hohenberg, 1993; Newell et al, 1993). One particularly important theoretical result, which motivated much of the work during the last decade, was that for a fluid of $\sigma \approx 1$ roll curvature induces slowly-varying long-range pressure gradients (Siggia & Zippelius, 1981) that drive a so-called "mean flow" which in turn couples back to the roll curvature. In subse-

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quent experimental work \(^3\) these ideas have found their convincing confirmation. Model equations which generalize the so called Swift-Hohenberg equations (SH-equations) (Swift & Hohenberg, 1977) were developed. They allow the isotropic description of the pattern-formation processes in the presence of mean flow (Manneville, 1983; Greenfield & Coughran, 1984). Although the SH-equations can not be derived systematically from the Boussinesq equations, they capture much of the observed physical behavior and have now become a general tool to investigate not only RBC, but also other pattern-forming systems (Cross & Hohenberg, 1993).

2 RECENT DEVELOPMENTS

Over the last 10 years the major experimental progress in RBC was due to the development of experimental apparatus for the study of convection of compressed gases in samples with large aspect ratios \(\Gamma = L/2d\). Here \(L\) is the typical lateral size of the convecting sample. Also very important was the rapid increase in computational power. The availability of large and fast storage media and of digital imaging equipment was crucial. It is now possible to study pattern dynamics by acquiring large time sequences of images. Some recent experimental projects have involved the analysis of over \(10^5\) high-resolution digital images (Hu et al., 1997).

At the end of the 1980s the new era started with the work by Croquette and coworkers (Croquette, 1989a,b), demonstrating that the horizontal planforms of thermal convection of compressed Argon gas could be visualized by the now well established shadowgraph technique (de Bruyn et al., 1996). Some very important mechanisms were identified and analyzed despite the use of rather small aspect-ratio cells (\(\Gamma \approx 15\)). In the meantime the aspect ratio of gas-convection cells has been pushed to much larger values (\(\Gamma > 100\)) (Bodenschutz et al., 1992b; Morris et al., 1993; Assenheimer & Steinberg, 1993). Typical experimental cells are constructed with a reflective bottom plate (silver, aluminum, aluminum coated sapphire, or silicon) and a transparent top sapphire plate. While the bottom plate is usually heated with a film heater, the top plate is cooled by a circulating water bath (de Bruyn et al., 1996). Two experimental designs have been used. In the first the top plate of the convection cell acts as a pressure window. This limits the size of the diameter due to unavoidable mechanical deformations by the large pressure differential which the plate has to sustain (Croquette, 1989a; Assenheimer & Steinberg, 1993; Assenheimer, 1994; Rogers et al., 1998). In the second the cooling water is pressurized, thus reducing the pressure differential to near zero. This allows the use of convection cells of larger diameter with little horizontal variations of \(R\) across the cell (de Bruyn et al., 1996)\(^4\).

Much of the theoretical work has been done in the Boussinesq approximation, which assumes that the fluid properties do not vary over the imposed temperature interval, except for the density where it provides the buoyant force. Experiments can be conducted under conditions which correspond closely to the Boussinesq approximation. A quasi-Boussinesq fluid is typically achieved by designing an experiment that yields a modest critical temperature difference \(\Delta T_c\) between the

\(^3\)See, for instance, Croquette et al. (1986b); Croquette (1989a); Daviaud & Pocheau (1989); Hu et al. (1995a); and Pocheau & Daviaud (1997).

\(^4\)Both \(\Delta T\) and \(d\) need to be constant across the cell, as \(R \propto \Delta T d^2\).
confining plates (typically not more than a few °C). This is achieved by using an appropriate cell height $d$. For conventional fluids like water, this implies that $d$ must be larger than about three $mm$. This in turn limits the maximum possible value of $\Gamma$ because uniform cells of excessive lateral extent are difficult to construct and expensive to produce. Another factor which prohibits the use of large lateral dimensions is the increase with aspect ratio $\Gamma$ of the horizontal diffusion time $t_h = \Gamma^2 t_v = \Gamma^2 d^2 / \kappa$ which influences the time required for pattern relaxation. For example, for an experiment with water as a Boussinesq fluid, the typical vertical diffusion time $t_v \approx 30 s$. For $\Gamma \approx 100$ this yields a horizontal diffusion time of $t_h \approx 84 h$. This would make experiments impractical, as some pattern-selection phenomena require time scales of tens of $h$ (Ahlers et al., 1985a).

Compressed gases have the advantage that the material parameters are such that a Boussinesq fluid can be achieved at rather small layer heights $d$. In typical experiments the layer heights are in the range from 0.3 to 1 $mm$, and $t_v$ can be one second or less. For a convection cell with $\Gamma = 100$ this gives a horizontal diffusion time of $t_h \approx 3 h$ (de Bruyn et al., 1996). For pure gases away from the critical point the value of the Prandtl number is $\sigma \approx 1$. However, it is possible to tune the Prandtl number in several ways. Assenheimer & Steinberg (1993, 1994) conducted experiments near the critical point of $SF_6$. They were able to cover the Prandtl number range $2 < \sigma < 30$. In their experiments the vertical thermal diffusion times ranged from $t_v = 3 s$ in a $d = 130 \mu m$ cell to $t_v = 27 s$ in a $d = 380 \mu m$ cell (Assenheimer, 1994). In another experiment Liu & Ahlers (1996, 1997) used gas mixtures to tune the Prandtl number from 0.17 < $\sigma$ < 1. For a cell of height $d = 1.46 \mu m$ their vertical thermal diffusion times were in the range 1.2s < $t_v$ < 6s. Thus in conclusion, compressed gases now allow experiments with large $\Gamma$ for 0.17 < $\sigma$ < 30.

The substantial progress described above has allowed a whole set of new experiments and has led to many new discoveries. For horizontal RBC Bodenschatz et al. (1991, 1992a) investigated the competition between hexagons and rolls in non-Boussinesq convection (see Sect. 6.5). They also discovered giant rotating spirals which were later analyzed in detail by Plapp & Bodenschatz (1996) and Plapp (1997) (see Sect. 6.3). In 1993 spiral-defect chaos (SDC) was discovered by Morris et al. (1993) in a parameter regime where on the basis of the theory for an infinitely extended system parallel straight rolls (ideal straight rolls or ISR) are stable (see Sect. 7). A great number of investigations followed that were concerned with the onset of SDC (see Liu & Ahlers, 1996, and references therein). Later in experiments by Cakmak et al. (1997a) it was demonstrated that both SDC and ISR are, in fact, stable attractors in the same parameter regime. These developments were accompanied by increased theoretical activities. SDC was reproduced in SH-model calculations (see Cross, 1996, and references therein) and in numerical solutions of the Boussinesq equations (see Pesch, 1996, and references therein) (see Sect. 3). From these investigations it is clear that SDC is a attractor which competes with ISR and that mean-flow effects due to roll curvature are important at not too large $\sigma$. In another set of experiments Assenheimer & Steinberg (1993) found that SDC evolved into a state of target chaos when the Prandtl number is raised (see Sect. 7). Assenheimer & Steinberg (1996) also found a state of coexisting up and down hexagons which were later explained theoretically by Clever & Busse (1996). During the last five years, experiments also studied the effect of rotation around a vertical axis (see Hu et al., 1998, and references therein), of inclination (Daniels et al., 1999), and of modulation of the
vertical acceleration (Rogers et al., 1998) (see Sect. 9). Another interesting topic, not discussed in this review, is thermal convection in planetary and stellar interiors, which is characterized by a finite angle between the vectors of rotation and gravity. This situation can be modelled in laboratory experiments through the use of centrifugal buoyancy (Busse et al., 1998). Time dependent complex states have been described but also new types of stationary patterns like the interesting hexa-rolls (Auer et al., 1995).

The following chapters will discuss the above mentioned developments in detail.

3 THEORETICAL ANALYSIS

In this section we sketch the common theoretical methods for the description of pattern-forming systems in order to emphasize the universal and common features between patterns in different systems. We exclusively consider large-aspect-ratio systems ($\Gamma \gg 1$), in which the horizontal dimensions of the fluid layer in the $(x,y)$-plane are assumed to be considerably larger than the cell height $d$. In typical experiments, shadowgraph visualization only shows the vertically averaged index-of-refraction variations.\footnote{To be precise, the shadowgraph signal is sensitive to the second derivative of the index of refraction with respect to the horizontal coordinates (de Bruyn et al., 1996; Berry & Bodenschatz, 1999).} This is analogous to theoretical treatments, where the interesting aspects of the patterns are reflected in suitable "projections" of the three-dimensional hydrodynamic description onto the two-dimensional horizontal plane. As discussed later in this section, such a dimensional reduction is reliable even for the complex spatial-temporal patterns presented in this review as the vertical spatial variations remain quite smooth.

Rayleigh-Bénard convection is described by the well known non-dimensional Boussinesq equations (see e.g., Busse (1978))

$$\sigma^{-1} \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \nabla^2 u + \hat{e}_z \Theta$$

$$\frac{\partial \Theta}{\partial t} + u \cdot \nabla \Theta = \nabla^2 \Theta + Re \hat{e}_z \cdot u$$

where $\hat{e}_z$ is the unit vector in the $z$ direction (opposite to the direction of gravity) and $\pi$ is the pressure. The velocity field $u$ and the deviation $\Theta$ of the temperature from the diffusive linear temperature profile vanish at the horizontal boundaries of the cell.\footnote{Alternatively, in the theoretical models free-slip boundaries have been used. One finds at small $\sigma$ and immediately above onset (chaotic) spatio-temporal chaotic patterns (Busse et al., 1992; Xi & Gunton, 1993).} Incompressibility ($\nabla u = 0$) is assumed. As already discussed in Sec. 1 the system is governed by two dimensionless quantities, namely the Rayleigh number $R$ and the Prandtl number $\sigma$. Equations (1, 2) show the sensitivity to $\sigma$. For $\sigma >> 1$ the left hand side of Eq. (1) can be neglected, thereby eliminating one of the nonlinearities. For $\sigma \approx 1$ both nonlinearities in Eqs. (1, 2) are important.

Equations (1,2) must be generalized for more complicated experimental situations, in which some of the underlying symmetries are broken. For instance, in the case of rotation about a vertical axis the Coriolis force $2\Omega \times u$ must be included in Eq. (1) (see e.g., Kippen & Lortz (1969)). In principle the centrifugal force should also be included; however, in experiments the rotation rate $\Omega$ is
typically kept so small that the centrifugal force is insignificant (see also Sec. 8). Convection becomes anisotropic when the layer is inclined, i.e., when the normal to the layer has a finite angle with gravity. In this case a shear flow exists already in the base state (see also Sec. 9.2).

In the most commonly used Boussinesq description the material parameters within the layer are approximated by their vertical average (again, except for the density which is responsible for the buoyancy force driving the convection). For experiments with large temperature gradients, vertical variations of the material parameters are significant and have to be included in the theoretical description (see eg., Busse (1967)). This leads to a transcritical bifurcation at threshold and to hexagonal patterns instead of rolls (see also Sec. 6.5). This situation is generally referred to as non-Boussinesq convection. In yet another variation the influence of temporal (see Sect. 9.3) and spatial modulations (Schmitz & Zimmermann, 1996) on RBC was considered.

3.1 Ideal straight rolls and their stability

The theoretical methods for analyzing pattern-forming instabilities are quite extensively discussed in the literature (see e.g., Cross & Hohenberg (1993), Newell et al (1993), and Pesch & Kramer (1995)). One expects, in good agreement with certain carefully-controlled experiments, that for large-aspect-ratio systems the simplifying idealization of an infinitely extended system is justified. This naturally leads to a convenient description of patterns in terms of Fourier modes in a two-dimensional (2d) wavevector (\( q \)) space. For an ideal straight-roll (ISR) pattern with wavelength \( \lambda = 2\pi/q \) only one pair of Fourier modes is excited; ideal square pattern require two pairs of modes, and ideal hexagons, three. These ideal patterns can be approximated experimentally in necessarily finite contain-
ers only under certain exceptional well-controlled conditions. For certain initial conditions, or when the influence of lateral boundaries is strong, experiments often produce disordered patterns which require a wave packet of many modes in their theoretical description.

The first step in the theoretical analysis is the standard linear stability analysis of the basic (primary) state. The problem diagonalizes in Fourier space with respect to the horizontal coordinates \( x = (x,y) \) according to the ansatz \( V(x,z,t) = e^{i\vec{q} \cdot \vec{x}} U(\vec{q},z) e^{i\lambda t} \). The symbolic vector \( V \) stands for the field variables \( \Theta, U \) in Eqs. (1.2). For fixed \( q \) one arrives at an eigenvalue problem with a discrete set of eigenvalues \( \lambda(\vec{q}, R) \) with the corresponding eigenvectors \( U(\vec{q},z) \), where the \( \lambda_{\vec{q} \in \mathbb{Q}} \) are ordered with respect to decreasing magnitudes of the real part. When, for increasing \( R \), the growth rate (i.e., \( \text{Re}(\lambda(\vec{q}, R)) \)) crosses zero at \( R = R_0(\vec{q}) \) (neutral curve) and becomes positive, the instability to ISR occurs. Minimization of \( R_0(\vec{q}) \) yields the critical wavevector \( \vec{q}_c \) and the threshold \( R_c = R_0(\vec{q}_c) \). ISR solutions of wavevector \( \vec{q} \) exist for \( R > R_0(\vec{q}) \geq R_c \). The bifurcation to ISR will either be stationary, i.e., when \( \exists \text{Im}(\lambda_1(\vec{q}_c, R_c)) = 0 \), or oscillatory (Hopf), i.e., when \( \exists \text{Im}(\lambda_1(\vec{q}_c, R_c)) \neq 0 \) as for instance at low \( \sigma \) in RBC with rotation (see Sec. 9.1). The \( \vec{q} \)-dependence of \( \lambda_1 \) (and correspondingly of \( R_0(\vec{q}) \)) reflects the spatial symmetries of the system. In rotationally invariant systems like RBC, the eigenvalues \( \lambda_1 \) depend only on \( |\vec{q}| \), whereas, for example, in axially anisotropic systems like the inclined layer (or nematic liquid crystals) the angle \( \phi_1 \) of \( \vec{q} \) with respect to the direction of anisotropy is also important: one observes, besides normal rolls (\( \phi_1 = 0 \)), situations with broken
chiral symmetry ($\phi_{\mathbf{q}} \neq 0$).

For our purpose a reformulation of the nonlinear partial differential equations, Eqs. (1,2), is very convenient (Pesch, 1996). The solutions $(\mathbf{u}, \Theta) \equiv \mathbf{V}$ are expanded in terms of the eigenvectors $U_i(\mathbf{q}, z)$ of the linear problem as

$$\mathbf{V}(x, z, t) = \sum_{i} A_i(\mathbf{q}, t) U_i(\mathbf{q}, z) \exp(iq x).$$

Projection on the $U_i$ leads to a coupled system of ODE's for the expansion coefficients $A_i$:

$$\partial_t A_i(\mathbf{q}, t) = \lambda_i(\mathbf{q}) A_i(\mathbf{q}, t) + N_i(A_i, A_j).$$

Note that the different $\mathbf{q}$-vectors are coupled by the projections $N_i$ of the quadratic nonlinearities in Eqs. (1,2). In practice, the equations are then suitably truncated ($i \leq N_{\text{cut}}$) and solved on a discrete mesh of wavevectors $\mathbf{q}$. In addition, it is convenient to capture the $z$-dependence of the $U_i$ by an expansion in suitable basis functions which fulfill the vertical boundary conditions ("Galerkin method", Clever & Busse (1974)).

The evaluation of periodic nonlinear stationary solutions $\mathbf{V}_{\text{stat}}$ of Eq. (4) amounts to the solution of a nonlinear set of algebraic equations; $\mathbf{V}_{\text{stat}}$ contains a basic mode with wavevector $\mathbf{q}_0$ together with its higher harmonics. As before, the stability of the stationary solutions is determined by an eigenvalue problem, which arises from the linearization of Eq. (4) around $\mathbf{V}_{\text{stat}}$. In general, one has to consider long-wavelength (modulational) and short wavelength instabilities. Perturbations with wavevectors $\mathbf{q} = n \mathbf{q}_0 + \mathbf{s}$, where $n = (0, \pm 1, \pm 2, \ldots)$, are considered and the stability is determined by searching the growth rates $\lambda_{\text{nonlin}}(\mathbf{q}, \mathbf{s}, R)$ for the largest real part. The corresponding eigenvector is dominated by a mode with wavevector $n_{\text{max}} \mathbf{q}$. In most cases there exists a region in $(R, \mathbf{q}, \lambda)$-space for which the periodic solutions are stable, i.e., where $\lambda_{\text{nonlin}} < 0$.

The boundaries of the stability region ("Busse balloon") are determined by various destabilization mechanisms (Busse, 1989). Of particular importance are the destabilizing long-wavelength modulations ($|\mathbf{s}| \ll |\mathbf{q}|$, $3m \lambda = 0$) of the ISR-pattern. One speaks of Eickhoffs (ECK) ($n_{\text{max}} = 1$ and $\mathbf{s} \parallel \mathbf{q}$, modulation of the roll distance), "zig-zag" (ZZ) ($n_{\text{max}} = 1$ and $\mathbf{s} \perp \mathbf{q}$, undulations along the roll axis), or the general case of "skewed varicose" (SV) instabilities ($n_{\text{max}} = 1$). For $\sigma \approx 1$, SV-instabilities delineate the boundary of the Busse balloon on the large-$\lambda$ side. For $\sigma < 1$ yet another instability, the oscillatory instability, limits the Busse balloon from above. This instability has $3m \lambda \neq 0$, $\mathbf{s} \perp \mathbf{q}$, and $n_{\text{max}} = 1$. In addition, short-wavelength instabilities can come into play. In this case the pattern is destabilized by disturbances with wavevectors $|\mathbf{s}| \approx |\mathbf{q}|$ but at a finite angle with respect to $\mathbf{q}$. For $\sigma \approx 1$ the "cross roll" (CR) instability with $\mathbf{s} \perp \mathbf{q}$ ($n_{\text{max}} = 0$, $3m \lambda = 0$) limits the Busse balloon at small wavenumbers. These instabilities are discussed in more detail for $\sigma \approx 1$ in Sec. 5. Experiment shows (see Egolf et al., 1998, and references therein), that to a good approximation, the idealized Busse balloon is also applicable locally to patches of ISR in an otherwise disordered pattern.

The K"uppers-Lortz instability (K"uppers & Lortz, 1969) occurs in RBC cells rotated around a vertical axis. Above a certain rotation frequency $\Omega_r(\sigma)$ periodic roll patterns are unstable, even at threshold. At large $\sigma$ one finds, for instance, $\angle(\mathbf{q}, \mathbf{s}) \approx 60^\circ$ with $|\mathbf{s}| \approx q_0$. 
3.2 Order parameter equations

Slightly above threshold and when \( \mathbf{q} \) is near \( \mathbf{q}_c \), the amplitude \( A_1(\equiv A) \) can be treated as the main dynamical variable, since only \( \Re(\lambda_1) \) is positive. By a systematic expansion up to cubic order in \( A \), which is assumed to be small near onset, one arrives at an order parameter equation (OPE) in Fourier space:

\[
\frac{\partial}{\partial t} A(\mathbf{q}, t) = \lambda_1(\mathbf{q}) A(\mathbf{q}, t) + \int dq_1 dq_2 A(q_1, q_2) A(q_1 - q_2, t) + \int dq_1 dq_2 dq_3 A(q_1, q_2, q_3) A(q_1 - q_2, t) A(q_3 - q_1 - q_2, t). \tag{5}
\]

The reduced dynamical description of patterns in terms of one amplitude \( A(\mathbf{q}) \) ("weakly nonlinear analysis") serves as the unifying description of many different pattern-forming phenomena. Systems differ mostly with respect to the explicit expressions for \( \lambda \) and the kernels \( a_2, a_3 \). The order-parameter formulation Eq. (5) is nowadays standard (Haken, 1978; Cross & Hohenberg, 1993; Newellet al., 1993). In particular, it has been shown that the OPE (5) can capture the Busse balloon even far away from threshold. Thus the OPE is particularly useful in describing systems where a fully nonlinear analysis is too demanding (Pesch & Kramer, 1995).

In RBC under Boussinesq conditions, where only the temperature variations of the density across the cell are kept, the \( a_2 \) term in Eq. (5) is zero. In this case, one obtains a supercritical (forward) bifurcation, i.e., above threshold the amplitude \( A(\mathbf{q}) \) grows continuously \( \sim \sqrt{(R - R_c)/R_c} \) (Schütter et al., 1965). Temperature variations of the material parameters like the viscosity or the thermal diffusivity are typically small and can be treated perturbatively in terms of a small parameter \( \mathcal{P} \) (Busse, 1967) (see also Cross & Hohenberg (1993), Sect. VIII.F.3). For \( \mathcal{P} \neq 0 \) the quadratic term \( (a_2 \propto \mathcal{P}) \) in Eq. (5) is nonvanishing and the symmetry \( A \to -A \) is broken. This reflects the broken inversion symmetry \( z \to -z, \Theta \to -\Theta, u_z \to -u_z \) due to additional terms in Eqs. (1,2). The onset of convection becomes transcritical towards a hexagonal pattern, which can exist for \( R \leq R_c \), since the quadratic term is activated due to a resonant coupling \( q_1 + q_2 + q_3 = 0 \) of the corresponding wavevectors (see Sec. 6.5).

Also important is that Eq. (5) can be taken as the general basis for a "derivation" of the reduced model equations for pattern forming systems. Let us briefly sketch the derivation of the Swift-Hohenberg (SH) equations for \( a_2 = 0 \) (Swift & Hohenberg, 1977). They can be obtained by transforming Eq. (5) to the 2D-position space for the amplitude \( \psi(x) = \int d\mathbf{q} \exp(i\mathbf{q}\cdot\mathbf{x}) A(\mathbf{q}) \). If the arguments of the kernel \( a_3(\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2) \) are suitably fixed at \( |\mathbf{q}| \), the cubic nonlinearity in Eq. (5) becomes proportional to \( \psi^3 \). Keeping additional terms from an expansion about \( \mathbf{q}_c \) leads to further cubic terms with gradients of \( \psi \). However, the expansion of the kernel is not smooth and certain nonanalytic contributions (the so-called "mean flow" terms) have to be treated separately in terms of a further amplitude \( B \) (Manneville, 1983; Cross & Hohenberg, 1993; Decker & Pesch, 1994). One arrives at:
\[
\partial_t \psi = \left(\epsilon - \frac{c_2}{4\xi_c} (\Delta + q_c^2)^2 \right) \psi - \psi^3 - \frac{1}{\xi_c} U \nabla \psi,
\]
\[
U = (\partial_y B, -\partial_x B),
\]
\[
(c_1 - \frac{c_2}{4\xi_c} \nabla^2) \nabla^2 B = \frac{\sigma_2^{-1}}{\xi_c} (\nabla (\nabla^2 \psi) \times \nabla \psi) \cdot \hat{e}_z.
\]

In Eq. (6) again \(\epsilon = (B - B_c)/B_c\) serves as the convenient measure of the distance from threshold. The coherence length \(\xi\) and \(c_1, c_2\) are constants. In some cases it is useful to further reduce the SH-equations by eliminating the fast spatial variations \(\propto q_c^{-1}\). One arrives at so called Ginzburg-Landau equations, such as the well-known Newell-Whitehead-Segel equation (Newell & Whitehead, 1969; Segel, 1989), which describe modulated patterns about a wavevector of a fixed direction. In the case \(B = 0\) (appropriate at large Prandtl-number) the dynamics in Eq. (6) is governed by a Lyapunov functional \(\mathcal{F}\) according to \(\partial \psi/\partial t = -\delta \mathcal{F}/\delta \psi\), and the time evolution is towards a fixed point. However, for finite \(B\) the system is not potential, thus opening the possibility for complex spatio-temporal behavior. Consequently, a description of patterns based on Eqs. (6) or their slight generalizations has become popular in analytical investigations as well as in many numerical studies (see Cross & Hohenberg, 1993). Based on Eqs. (6) researchers have successfully investigated the dynamics of defects and grain boundaries (see Cross & Hohenberg, 1993), the stability of planforms and transitions between them (e.g. from hexagons to rolls (Xi et al., 1992)), and the influence of noise (see Sec. 4). In particular, important insight into the mechanism of spiral-defect chaos (SDC) (Xi et al., 1993; Bestehorn et al., 1992) and the Kellers-Lortz instability (Tu & Cross, 1992a) has been achieved on the basis of the SH-formulation. One should note, however, that due to missing higher-order derivative terms the Busse balloon is in general not described systematically (Decker et al., 1994).

3.3 Phase-diffusion equation

Even far above onset, the structure of periodic patterns far away from defects and grain boundaries can be captured by nonlinear phase-diffusion equations. These equations describe slow variations of the orientation and spacing of the convection rolls (see Newell et al., 1993, and references therein). Let a family of stationary, periodic, and reflection-symmetric solutions be denoted by \(V(q, x) = u_q(q \cdot x)\) with \(u_q(\phi + 2\pi) = u_q(\phi)\) and let us write the actual state as \(\tilde{V}(x, t) = u_q[\phi(x, t)] + \) corrections. Thus \(\phi(x, t)\) is the phase of a nearly-periodic pattern and \(\tilde{V}(x, t) = q(x, t)\) plays the role of the local wave vector\(^7\), which we assume to vary slowly in space and time. In a seminal paper, Pomeau & Manneville (1979) showed that the phase dynamics can be described by a diffusion equation. They also found that at the long-wavelength instability boundaries of the Busse balloon certain diffusion constants changed sign.

Subsequently, a rotationally invariant form of the phase equation which also included the mean flow \(U\), the Cross-Newell equation, was derived (see Newell et al., 1993, and references therein). In its simplest from the equation reads\(^7\)

\(^7\)In a real field the wavevector \((q_x, q_y)\) is equivalent to \(-q_x, -q_y\), i.e., \(q(x)\) is actually a director field.
\[
\tau(q) \frac{\partial}{\partial t} \varphi + \mathbf{U} \mathbf{q} = \nabla \cdot (B(k) \mathbf{q}) \tag{7}
\]
\[
\text{curl}_z \mathbf{U} = -\gamma \sigma^{-1} \mathbf{e}_z \cdot \nabla \times \left[ \mathbf{q} \nabla \cdot \left( \mathbf{q} A^2 \right) \right]. \tag{8}
\]

Phase equations are very valuable in the analysis of non-ideal periodic patterns. This applies, for instance, to the dynamics of dislocations in an ISR pattern. In certain relatively simple patterns, dislocations climb with a velocity \( v = \alpha(q - q_0) \) where \( \alpha \) is almost constant and \( q \) is the wavenumber of the pattern (Tessaro & Cross, 1986). Consequently a dislocation is stationary for a roll pattern with background wavenumber \( q = q_0(\varepsilon) \). Also using this approach, Newell et al. (1991) determined a wavenumber \( q_f(\varepsilon) \), \( q_f(0) = q_c \), which is selected by concentric patterns. Those authors demonstrated as well that at small \( \sigma \) concentric patterns become unstable at a certain quite small value of \( \varepsilon \) due to the coupling to the mean flow. It has been suggested by Cross & Tu (1995) and shown by Plapp et al. (1998) that the competition of the two selected wavenumbers \( q_f \) and \( q_d \) determines the dynamics of the giant spirals, as discussed in Sect. 6.3.

The possible defect structures in natural convection patterns, such as dislocations or grain boundaries, correspond to singularities in a two-dimensional (wave) vector field, and are well-classified in differential geometry. Recently this insight has considerably improved the understanding of the solution manifold of the phase-diffusion equations (Newell et al., 1996; Bowman et al., 1998; Ercolani et al., 1999) Although the underlying basic hydrodynamic description of convection patterns is nonpotential, the phase equations often derive from a Lyapunov functional (Newell et al., 1996). Whether such a potential could provide some general selection principle according to which natural patterns approach stationary configurations is at the moment an open question.

Figure 1: Radial roll structure for an annulus for \( \sigma = 1.0 \) at \( \varepsilon = 0.4 \). (I.V. Melnikov and W. Pesch, unpublished).
3.4 Numerical simulations

The (large) system of ODE's in Eq. (4) has become very useful for the generation of numerical solutions of the Boussinesq equations (Decker et al., 1994; Pesch, 1996). In contrast to the standard discretization schemes the use of Eq. (4) permits a description of the dynamics in terms of the most important active modes ($\Re(\lambda)$ not too small). Passive modes can be adiabatically eliminated. In the numerical implementation the most time consuming manipulations can be based on the fast Fourier transformations (FFT). The speed of this pseudo-spectral method permits the simulation of the large-aspect-ratio systems used in experiments. In the time-stepping scheme the linear operator is treated fully implicitly, whereas an explicit Adams-Bashforth scheme is used for the nonlinear part. A time step as large as $0.02t_0$ yields stable performance. One must keep in mind that diffusive changes of the patterns take place on much longer time scales ($\Gamma^2t_0$). However, simultaneously one must resolve processes on time scales of order $t_0$, such as the nucleation or annihilation of defects or the fast core rotation of spirals. The code can also be used to approximate non-periodic boundaries by the use of suitable spatial ramps in the Rayleigh number. An example is given in Fig. 1. The simulation was started from random initial conditions and shows the pattern after transients had died out. Apparently it is not difficult to reproduce the common experimental situation where the rolls meet the boundaries perpendicularly.\footnote{This is not to imply, however, that this simple spatial variation of the control parameter is in any sense equivalent to the physical lateral boundary conditions relevant to real experiments.}

Further examples of simulations are shown in the forthcoming sections (see also Pesch (1996)). Another advantage of the direct simulations is the ability to use an experimental shadowgraph picture as an initial condition and then reconstruct the entire 3d convective structure corresponding to the experiment (see Sec.7).

4 FLUCTUATIONS

As described in Sec. 3, the study of convection patterns is usually based on a stability analysis of the deterministic hydrodynamic equations. Such a theory gives a sharp threshold at $R = R_c$, with $u$ and $\Theta$ equal to zero below it (see Figs. 1 and 2). For $R > R_c$, or equivalently $\epsilon > 0$, convection rolls with wave number $q_0$ are predicted to initially grow exponentially in time and thereby to destabilize the motionless state. One has to keep in mind, however, that the "thermal noise" of the microscopic degrees of freedom has been averaged away in this treatment. This noise drives fluctuations of the macroscopic velocity and temperature fields about their mean values even below the bifurcation. For a constant noise intensity and in the absence of nonlinear saturation, the amplitudes of these fluctuations diverge as the bifurcation point is approached. For this reason, the deterministic approach breaks down in the close vicinity of onset ($\epsilon \approx 0$). This is analogous to second-order phase transitions in equilibrium thermodynamics, where fluctuations become large close to the critical temperature.

This problem received considerable theoretical attention about three decades \footnote{This work was presented at the workshop Spatiotemporal Characterization of Spiral Defect Chaos at Los Alamos National Laboratory, January 4-5 1999. Sensoy & Greenside (1999) investigated similar structures within the SI-equations.}
ago (for a review see e.g. Cross & Hohenberg, 1993, Sect. VIII.D.1.c). The analysis was based on the stochastic Navier-Stokes equation given by Landau & Lifshitz (1959) and on stochastic Ginzburg-Landau and Swift-Hohenberg equations. Time-dependent fluctuating convective flows with zero mean but finite mean-square amplitudes $< A^2 >$ were predicted for $\epsilon < 0$. At $\epsilon = 0$, $< A^2 >$ diverges in proportion to $(-\epsilon)^{-1/2}$ until nonlinear saturation sets in. However, estimates of the noise strength suggested that the fluctuations should remain unobservably small at experimentally accessible values of $\epsilon$ because the thermal energy $k_B T$ ($k_B$ is the Boltzmann constant) which drives them is many orders of magnitude smaller than the typical kinetic energy of a macroscopic convecting fluid element (see e.g. Swift et al, 1991; Hohenberg & Swift, 1992). Nonetheless, it has now become possible to observe the fluctuating convection patterns below the bifurcation directly and to make quantitative measurements of their root-mean-square (rms) amplitudes.

The first system for which this became possible was electroconvection in a nematic liquid crystal (Rehberg et al, 1991). Even though that system is "macroscopic", it is particularly susceptible to noise because the physical dimensions are only of order $10 \mu m$ and because the elastic constants (which determine the macroscopic energy to which $k_B T$ has to be compared) are exceptionally small.

![Image](image_url)

Figure 2: Left: Shadowgraph snapshot of fluctuations below the onset of convection ($\epsilon = -3 \times 10^{-4}$). Right: The average of the square of the modulus of the Fourier transform of 64 images like that on the left. After Wu et al (1995).

More recently, fluctuations were observed also for RBC by Bodenschatz et al (1992a), and quantitative measurements of their amplitudes were made by Wu et al (1995). These measurements were made possible by maximizing the sensitivity of the shadowgraph method (de Bruyn et al, 1996) and by careful digital image analysis. The left part of Fig. 2 shows a processed image of a layer of CO$_2$ of thickness $0.47 mm$ at a pressure of 29 bar and at a mean temperature of $32.0^\circ C$. The sample was at $\epsilon = -3 \times 10^{-4}$, very close to but just below the bifurcation.

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10Measurements on binary-mixture convection by Schöpf & Rehberg (1994) and by Quentin & Rehberg (1995) involved pseudo-one-dimensional sample geometries for which theoretical predictions are more difficult to obtain due to the influence of the sidewalls.
point. The fluctuating pattern is barely detectable by eye. The right half of the figure shows the average of the structure factors (squares of the moduli of the Fourier transforms) of 64 such images. Clearly demonstrating that the fluctuations have a characteristic wavenumber $q$ which was found to be quantitative agreement with the critical wavenumber $q_c = 3.117$ for RBC. The ring in Fourier space is azimuthally uniform, reflecting the continuous rotational symmetry of the RBC system.

The power contained within the ring in Fourier space can be converted quantitatively to the mean square amplitude of the temperature field (Wu et al., 1995; de Bruyn et al., 1996). Results for the temporal and spatial average $\delta T^2(\epsilon)$ of the square of the deviations of the temperature from its local time average (pure conduction) as a function of $\epsilon$ at two different sample pressures are shown in Fig. 3. The data can be described quite accurately by the powerlaw $\delta T^2 \propto \epsilon^{-1/2}$, as predicted by theory.

![Graph showing mean square amplitudes of temperature fluctuations](image.png)

**Figure 3**: Mean square amplitudes of the temperature fluctuations below the onset of convection of a layer of CO$_2$ of thickness 0.47 mm and a mean temperature of 32°C. The solid (open) circles are for a sample pressure of 42.3 (29.0) bar. The two lines are the theoretical predictions. Note that there are no adjustable parameters. After Wu et al. (1995).

The amplitudes of the fluctuating modes below but close to the onset of RBC were calculated quantitatively from the stochastic hydrodynamic equations (Landau & Lifshitz, 1959) by van Beijeren & Cohen (1988), using realistic (no-slip) boundary conditions at the top and bottom of the cell. For the mean square temperature fluctuations their results give (Hohenberg & Swift, 1992; Wu et al., 1995)

$$\delta T^2(\epsilon) = \bar{\epsilon}^2 \left( \frac{\Delta T_c}{R_c} \right)^2 \frac{F}{4\sqrt{-\epsilon}} ,$$

with $\bar{\epsilon} = 3q_e\sqrt{R_c} = 385.28$ and

$$F = \frac{k_B T}{\rho c \nu^2} \times \frac{2\sigma q_c}{\xi_0 \tau_0 R_c} ,$$

with $\xi_0 = 0.385$ and $\tau_0 = 0.0796$. Using the fluid properties of the experimental samples, one obtains the straight lines in Fig. 3. The agreement between theory and experiment is very good. This agreement lends strong support to the validity of Landau’s stochastic hydrodynamic equations (Landau & Lifshitz, 1959).
Sufficiently close to the bifurcation the fluctuations should induce deviations from the usual mean-field behavior implied by Eq. (9) (Hohenberg & Swift, 1992). In the case of Rayleigh-Bénard convection, this deviation is predicted to take the form of a hysteric bifurcation (Hohenberg & Swift, 1992; Brazovskii, 1975). However, this should become observable only in the range \( |\epsilon| \lesssim 10^{-6} \) or so, which is not expected to become experimentally accessible in the near future.

Figure 4: Straight parallel rolls for circular and square sidewalls. (a) From Liu & Ahlers (1996) at \( \epsilon = 0.07 \) with \( \Gamma = 30 \) and \( \sigma = 0.69 \). (b) After Calmure et al (1997a) at \( \epsilon = 0.3 \) with \( \Gamma = 50 \) and \( \sigma = 1.03 \).

5 IDEAL STRAIGHT ROLLS

Above but close to onset the predictions of the linear and weakly-nonlinear theory (Schlüter et al, 1965) were reproduced quantitatively in a number of experiments (see Cross & Hohenberg, 1993, and references therein). A particularly detailed study was carried out by Hu et al (1993). For cases where the fluid properties are known sufficiently well, measurements yielded values of \( R_c \) within a few percent of the theoretical value, \( R_c = 1708 \) (de Bruyn et al, 1996). Experiments with quasi-Boussinesq fluids (see Sect. 3) yielded a supercritical bifurcation at \( R_c \). Above onset, almost defect-free roll patterns with wavenumbers close to the theoretical value \( q_c = 3.117 \) were found. Examples are shown in Figs. 4a and b for a circular and a square cell, respectively. Beyond \( R_c \) the amplitude of the shadowgraph signal (Fig. 8 of Hu et al (1993)), proportional to the amplitude of the temperature field \( \Theta \), grows as \( \sqrt{\epsilon} \), as is evident, for instance, from the open circles in Fig. 12 below in Sect. 6.2. The convective heat transport, usually expressed as the Nusselt number \( N \) (the ratio of the effective conductivity of the fluid to the conductivity in the absence of convection), can be written as \( N = S_1 \epsilon \) for small positive \( \epsilon \) and shows no hysteresis at the bifurcation (Fig. 4 of (Hu et al, 1993)). However, \( S_1 \) usually is somewhat smaller than the prediction (Schlüter et al, 1965) for the laterally infinite system. The reason for this may be that the flow near the walls, as well as near any defects that may be present, is suppressed.

At values of \( \epsilon \) larger than about 0.1, the rolls in circular cells develop strong curvature and defects appear in the patterns. These interesting boundary-induced
Figure 5: (a) ISR and (b) SDC at $\epsilon = 0.92$; (c) ISR at $\epsilon = 2.99$ and (d) SDC at $\epsilon = 3.0$; (e) oscillatory ISR and (f) oscillatory SDC at $\epsilon = 5.08$. For this experiment $\Gamma = 50$ and $\sigma = 1.03$. For each pair of pictures only the initial conditions were different. The insets show a magnified view of the oscillating rolls. While in (e) the oscillations travel from bottom to top, in (f) the oscillations are very disordered. Often rotating spoke pattern are found as seen in the insert of (f). From Cakmuh et al (1997a).
phenomena will be discussed in Sect. 6.1. Due to their geometry, rectangular cells are better suited for the study of parallel straight rolls over a wide \(\varepsilon\)-range. Patterns can be created which come close to the theoretical idealization of ISR, and they persist even at large \(\varepsilon\). In the remainder of this section, we review investigations of the Busse balloon and its limiting instabilities for ISR in cells with square geometry.

![Diagram](image)

Figure 6: The stability boundaries of ISR for \(\sigma = 1.03\) with experimental data. The theoretical curves are denoted SV (skewed varicose), CR (cross roll), ECK (Eckhaus), and OSC (Oscillatory). The arrows indicate the path taken while increasing \(\varepsilon\). Open circles before SV-instability; upside-down triangles after SV-instability; diamonds onset of OSC-instability; squares localized CR-instability for decreasing \(\varepsilon\); solid circles numerically determined boundary for localized CR-instability; triangles wavenumber of Spiral Defect Chaos as measured at the maximum of the azimuthally averaged powerspectrum. From Plapp (1997).

For \(\sigma \approx 1\) Figs. 5(a,c,e) demonstrate that stable ISR patterns can be observed experimentally over a wide range of \(\varepsilon\) (Cakmur et al., 1997a; Plapp, 1997). Even above the OSC-instability boundary (Fig. 5e) the pattern was ISR-like. However, ISR only formed when a perfect parallel-roll pattern was initially prepared by special procedures (described below). The generic attractor starting from random initial conditions is a spatio-temporal chaotic state called spiral-defect chaos (SDC) (Morris et al., 1993) which is discussed in detail in Sect. 7. In Fig. 5 images of SDC are compared with ISR at the same \(\varepsilon\)-values.

Ideal straight rolls were initialized in a square cell of \(\Gamma = 50\) by a special procedure.\(^{11}\) The protocol involved inclining the cell to induce a large-scale flow which aligned the rolls (Cakmur et al., 1997a; Plapp, 1997; Jeanjean, 1997). After appropriate equilibration procedures, the rolls could then be used to explore the Busse balloon. The stability boundaries were determined by suitable quasistatic changes of \(\varepsilon\) as indicated by the arrows in Fig. 6. When the SV-boundary at the

\(^{11}\) Only large wavenumbers close to the skewed-varicose stability boundary were stable, otherwise cross roll disturbances grew at the boundaries leading to SDC.
Figure 7: Time evolution of the skewed-varicose instability at $\epsilon = 2.26, \sigma = 1.07$ (about 7.5 minutes ($183t_v$) after $\epsilon$ was increased from 2.23). Pictures are spaced $0.54t_v$ apart. From Plapp (1997).

High wavenumber was crossed (circles), the SV-instability occurred and nucleated one or two defect pairs by the well known “roll pinching” (Croquette, 1989a). As shown in Fig. 7, the defects traveled along the roll axis to the boundaries, destroying one or two roll pairs and thus changing the wavenumber back into the stable regime (triangles). This way Cakmur et al. (1997a) were able to follow the SV-instability-boundary until they crossed the oscillatory instability-boundary (OSC). As shown in Fig. 6, good agreement between the theoretical predictions for the laterally infinite system and the experiment was found.

Above the oscillatory instability line (OSC), a triangular traveling-wave pattern is superimposed onto the rolls (Busse, 1972; Croquette & Williams, 1989a,b). This can be seen already in Fig. 5e, and an example is shown for left-traveling waves in Fig. 8A. Fourier-demodulation techniques are well suited for a detailed investigation of such patterns (Rosenat et al., 1990; de Bruyn et al., 1996). While Fig. 8B shows the power spectrum, Figs. 8C and 8D show the real part of the Fourier back-transform of the areas encircled by the black and gray lines, respectively. The spatial disorder of the traveling waves is most evident in the demodulated pattern shown in Fig. 8D. The oscillatory instability is convective (Croquette & Williams, 1989b; Babcock et al., 1994). Indeed, as shown in Fig. 8D, the amplitude of the left-traveling waves is observed to increase “down stream”. Consequently, it is not surprising that the measured onset of the oscillatory instability for $\sigma = 1.03$ was found to be slightly larger than the theoretical one: growing fluctuations have not enough time to reach visible amplitudes, before they are absorbed at the lateral walls.

When $\epsilon$ was decreased quasistatically starting from the oscillatory regime, the ISR-pattern had a tendency to increase its wavenumber by contracting and nucleating cross rolls at the boundaries which were parallel to the rolls (Cakmur et al., 1997a). Spiral-defect chaos then nucleated in the grain boundaries and the ISR-state was destroyed (see Sect. 7). It was still possible, however, to measure the CR-instability-boundary by decreasing $\epsilon$ sufficiently rapidly so as to avoid pattern relaxation in the bulk. Surprisingly, three different nonlinear evolutions
of the CR-instability were found. As shown in Fig. 9A for $\epsilon = 0.80 \pm 0.15$, a cross-roll defect nucleated at one of the sidewalls and propagated in a direction which increased the wavenumber. For $\epsilon = 0.57 \pm 0.15$ the cross-roll defect left behind a disordered “topology” like pattern while moving through the system (Fig. 9B). For $\epsilon = 0.25 \pm 0.10$ the CR-instability occurred in the bulk (Fig. 9C).

These local CR-instabilities were also found in numerical simulations when, for fixed $\epsilon$, an ISR-pattern was initialized with two oppositely charged dislocations and a wavenumber close to the CR-boundary. The results of the simulations are as shown by the solid circles in Fig. 6. In the simulations the totem pole pattern leads to the nucleation of SDC.

While the bulk instability is similar to the one observed for larger Prandtl number fluids (Buse & Whitehead, 1971), the other two are localized CR-instabilities. Localized CR-instabilities were first observed by Croquette (1989a); however, due to the small aspect ratio used in the experiment it was not possible to study the full dynamics. The localized CR-instability has also been called a bridging instability (Newell & Pasol, 1992; Assenheimer & Steinberg, 1994) and has been associated with a different nonlinear effect.
6 CIRCULAR CELLS

6.1 PanAm patterns

As discussed above, close to onset pseudo-ISR are stable in circular as well as in rectangular geometries. However, for circular cells there is a tendency to form short cross rolls near that part of the sidewall where the rolls would otherwise be parallel to the wall (see Fig. 4a). The cross-roll patches, separated by grain boundaries from the bulk, are a manifestation of an often observed tendency for rolls to terminate with their axes perpendicular to the sidewalls. This orientation at the boundaries is also typically observed in numerical simulations without forcing lateral boundary conditions (see e.g. Fig. 1). Loosely speaking this configuration minimizes the friction experienced by the rolls at the walls. With increasing $\epsilon$ this effect becomes more pronounced. In circular convection cells it leads to enhanced roll curvature in the pattern interior. As the curvature increases, the domain walls shrink and form focus singularities at the wall (wall foci). In relatively small aspect-ratio systems typically two wall foci form, and then for obvious reasons the resulting structures often are referred to as “PanAm” patterns. For larger $\Gamma$, however, three or more wall foci can form as $\epsilon$ increases, and more complicated structures arise. An example of a PanAm pattern is given by the smallest sample shown in Fig. 10B below. More complicated textures are shown in Fig. 10A. While this situation is generic for quasi static increase of $\epsilon$,
transient sidewall forcing can also select target patterns (see Sect. 6.2) which (in the presence of some static sidewall forcing) are stable in the same $\epsilon$ range as the curved rolls with wall foci. This is illustrated by the larger samples in Fig. 10B.

Figure 10: An example of (A) PanAm ($\epsilon = 0.34, \sigma = 1.39$) and (B) target patterns ($\epsilon = 0.38, \sigma = 1.40$) in the same convection cells with $\Gamma = (48.4, 38.8, 28.4, 23.8, 19.1, 14.9)$ for different experimental paths. From Plapp (1997).

The competition between curvature and sidewall obliqueness was examined theoretically by Cross (1982). A functional of the wavevector field was derived which had contributions from wavenumber variations, from roll curvature, from sidewall obliqueness, and from defects. The minimization of this functional drives the selection of the pattern. Semi-quantitatively these predictions were confirmed by experiments with $\sigma \gg 1$ (Heutmaker et al., 1985; Heutmaker & Golub, 1986, 1987). However, for $\sigma \approx 1$ mean-flow effects are important. Indeed it was shown experimentally by Daviaud & Pocheau (1989) that suppression of the mean flow dramatically reduces the roll curvature. When large-scale flows have a strong influence on the pattern, the applicability of a model with a potential seems in question even for relatively small $\epsilon$-values.

The stability of ISR is described by the Buse balloon (see Sects. 3 and 5). The stable states are limited at the low wavenumber side by the ECK- and the CR-instabilities and at the high wavenumber side by the SV-instability. An important, as yet largely unanswered, question is which of the continuum of stable states will be selected by the physical system. There is no known extremum principle. Instead it appears that preferred wavenumbers are chosen by a given set of boundary conditions, defects, and/or histories. For example, for a circular cell of aspect ratio $\Gamma \geq 60$ the domain walls separating the cross rolls from the main rolls permit a continuous wavenumber adjustment in the cell interior. It was found experimentally for circular cells (Hu et al., 1993; J. Lin, K.M.S. Bajaj, G. Ahlers, unpublished) that this leads to the selection of a unique wavenumber as shown by the plusses and open circles in Fig. 11. When $\epsilon$ reaches a value of about 0.1, the selected wavenumber is close to the SV-instability. Indeed, the roll curvature which prevails at this point (wall foci developed) leads to a significant wavenumber distribution, with the largest $q$ in the sample interior. These largest wavenumbers cross the SV-instability at $\epsilon \approx 0.11$. Then defect nucleation begins
in the cell interior, as noted by Croquette (1989b). Although the elimination of a roll pair by the defect creation and migration moves the system back into the stable interior of the Busse balloon, the wall foci emit new rolls and a time dependent state persists (Croquette, 1989b; Hu et al, 1994, 1995a). This will be discussed further in Sect. 6.4.

6.2 Targets.

Even in very carefully constructed experimental cells, the existence of sidewalls can introduce horizontal thermal gradients near them. This can have a surprisingly strong impact on the pattern-formation processes (see e.g. Cross & Hohenberg (1993) Sec. VIII.D1, and de Bruyn et al (1996)). The gradients have the tendency to drive flow fields which have the symmetry of the walls. Static (time independent) forcing occurs, for instance, when the conductivity of the walls is significantly different from that of the fluid and the conductivity of the top or bottom plate is finite, albeit large. In some experiments, static forcing was induced deliberately by embedding a heater in the sidewall (see, for instance, Croquette (1989a) and Morris et al (1996)). Sidewall forcing can also be dynamic due to a mismatch of the thermal diffusivity between sidewall and fluid. In that case, a change in, for instance, the bottom-plate temperature will cause vertical temperature profiles which relax at different rates in the wall and in the fluid, again leading to (in this case transient) horizontal gradients near the wall.

In circular cells sidewall forcing (transient or static) can lead to concentric patterns, also known as targets, like those in Fig. 10B. Such patterns have been investigated in numerous experiments. They are found to be stable for small

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\( \varepsilon \) until, with increasing \( \varepsilon \), an instability known as the focus instability (Newell et al, 1990) occurs at their center.

![Figure 12: The amplitude of the umbilicus (solid circles), and of the rolls away from the center (open circles), in a circular cell with \( \Gamma = 43 \) as a function of \( \varepsilon \) on logarithmic scales. After Hu et al (1993).](image)

An interesting aspect of concentric patterns in their stable range (below the focus instability) is an anomalous variation of the amplitude \( A \) of their umbilici, i.e., target centers, which was noted qualitatively by Croquette et al (1986a). This is illustrated by the solid circles in Fig. 12, which represent more recent quantitative measurements (Hu et al, 1993) of \( A^2 \) as a function of \( \varepsilon \) on logarithmic scales. The data can be fit by a line with a slope of \( \frac{1}{4} \), indicating that \( A \propto \varepsilon^{1/4} \).

This can be compared with the dependence of the roll amplitude well away from the center of the pattern, which is shown as open circles and which is consistent with \( A \propto \varepsilon^{1/2} \) as predicted on the basis of the Landau equation for a supercritical bifurcation. The unusual behavior of the umbilicus amplitude occurs because the Landau equation is not applicable (Brown & Stewartson, 1978; Ahlers et al, 1981) at the pattern center, as was explained by Pouget et al (1985) who predicted the exponent value \( \frac{1}{4} \) in the limit of a large-aspect-ratio cell.

The events which occur as \( \varepsilon \) is increased beyond the focus instability depend on the strength of the sidewall forcing, the wavenumber, the aspect ratio, and the Prandtl number. They have not yet been predicted by theory. One set of events observed in experiments is that the umbilicus drifts away from the center and toward the wall, thus destroying the concentric pattern (Hu et al, 1993). Another involves the periodic emission of outward-traveling concentric rolls (Hu et al, 1993). Yet another observed scenario is that the umbilicus moves slightly off-center, and then reestablishes at a smaller pattern wavenumber by destroying one concentric roll pair near it (Croquette et al, 1983; Steinberg et al, 1985). A further mechanism involves an off-center displacement of the umbilicus which is followed by radial oscillations of the umbilicus position (Plapp, 1997).

When the focus instability leads to a change in the number of concentric convection rolls and the sidewall forcing is sufficiently strong, the concentric nature of the pattern can be maintained over a large \( \varepsilon \) range and the umbilicus provides a wavenumber-selection mechanism.\(^{15}\) The selected wavenumber depends

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Figure 13: (a) to (l): birth of a one-armed clockwise-rotating spiral for $\Gamma = 43$ and $\sigma = 1.38$ after $\epsilon$ was increased from 0.58 to 0.60. Pictures (a) to (h) are spaced $0.5t_0$, apart, and (h) to (l) are at intervals of $1.5t_0$, (m) to (p): a rotating single-armed spiral. The pictures are spaced $82t_0$, or a $1/4$ period, apart. From Plapp (1997).

significantly on the Prandtl number, and can be larger (small $\sigma$) or smaller (large $\sigma$) than $q_0$. It was calculated by several authors (Manneville & Piqueau, 1983; Cross, 1983; Boell & Catton, 1986a), and there is generally good agreement with the measurements.

6.3 Giant spirals

Multiarmed, giant, rotating spirals were observed in a number of experiments with $\sigma \approx 1.16$. They were discovered by Bodenschatz et al (1991), and studied in detail by Plapp and coworkers (Plapp & Bodenschatz, 1996; Plapp, 1997; Plapp et al, 1998) who investigated their formation, dynamics, selection, and stability. Multi-armed spirals can be obtained by appropriate jumps in $\epsilon$ or by increasing $\epsilon$ slowly. Some static sidewall forcing seems required to prevent the center of the large structure from drifting toward the sidewall. One example of spiral creation is illustrated in Fig. 13. In this case, the umbilicus of an initial target moved

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Figure 14: (a): Snapshot of a three-armed spiral for $\Gamma = 388$, $\epsilon = 0.79$, and $\sigma = 1.37$. (b): average of images like that in (a), covering one period of rotation. (1) to (12) show the fast core rotation over one period. Here frames are spaced $0.44t_0$ apart. From Plapp & Bodenschatz (1996) and Plapp (1997).

off-center after $\epsilon$ was increased quasi-statically. Compression of the rolls in the lower right part of the pattern increased the local $q$ beyond the SV-instability and caused a dislocation pair to be formed. One of the dislocations moved to the center of the target while the other remained at a finite radius from it, thus creating a one-armed spiral.

Figure 15: Wavenumber adjustment of a one-armed spiral for $\Gamma = 28.4$ and $\sigma = 1.38$. (a) $\epsilon = 0.55$ and (b) $\epsilon = 0.72$. Note the position of the outer defect in relation to the core. From Plapp (1997).

$N$-armed spirals rotate slowly by emitting radially-traveling waves which are annihilated by $N$ dislocations which rotate synchronously with the spiral heads (Bodenschatz et al., 1991, 1992b). Beyond the dislocations the spiral arms are surrounded by a target pattern. The core of a $N$-armed spiral consists of $N$ bound dislocations of equal topological charge (Plapp et al., 1998). By time averaging
a giant spiral over one rotation period it is possible to recover the underlying
target pattern (Plapp et al, 1998). This is shown for the example of a three-
armed spiral in Fig. 14. Also shown is the fast core rotation that can be observed
for multi-armed spirals (Plapp & Bodenschatz, 1996; Assenheimer, 1994).

With increasing $\epsilon$, giant spirals can adjust their wavenumber by changing the
position of the core with respect to the revolving dislocation (Plapp, 1997). The
spiral either winds up or unwinds to adjust its wavenumber, as illustrated in
Fig. 15. Once the wavenumber is adjusted, the spiral's tip at the core is locked in
relation to the outer defect. In this way the spiral can decrease its wavenumber
continuously with increasing $\epsilon$, as shown in Fig. 17.

Figure 16: Two-armed spiral patterns in cell 1 and 4. Spiral defect chaos in cell
2. Textures with wall foci in cell 3 and the unmarked cells. For these images
$\epsilon = 0.98$ and $\sigma = 1.4$. From Plapp et al (1998).

As $\epsilon$ is increased further, the spiral structures become unstable (Plapp, 1997;
Plapp et al, 1998). Typically the cores move off center and cause a local increase of
$q$ beyond the SV instability. The pattern develops many dislocations. Depending
on parameters, the structure evolves into a many-armed spiral, a texture with wall
foci, or spiral-defect chaos. Interestingly, as shown in Fig. 16, bistability between
rotating spirals and the spatio-temporal chaotic planforms of spiral defect chaos
and PanAm-like patterns were observed.

The rigid rotation of a stable giant spiral requires that the radial waves which
propagate from the center are annihilated at a radius $r = r_d$ by the circular motion
of the outer defect. Thus for $r > r_d$ a concentric stationary roll pattern results.
Arguments based on the phase-diffusion equation (see Sec. 3.3) can be used to
quantitatively describe the large-scale rotation of the spirals. It was proposed
(Cross & Tu, 1995; Cross, 1996; Li et al, 1996) that the rotation of giant spirals
requires the reconciliation of two competing wavelength-selection mechanisms.
Figure 17: Wavenumbers selected by targets, spirals, and dislocations for a circular cell of $\Gamma = 38.8$ for $\sigma = 1.4$. The region marked $q_d$ gives the experimentally (squares) and theoretically (triangles and dashed line) determined dislocation-selected wavenumbers. The open circles show the target-selected wavenumber $q_t$ and the solid line is the extrapolation to larger $\epsilon$. The square symbols marked 1, 3, and 4 represent the selected wavenumbers of one, three, and four-armed spirals. Also shown is the Buse ballon. After Plapp (1997).

acting far away from the spiral’s core, namely wavelength selection by defect climb and target wavelength selection by radially-travelling waves. Tisamo & Cross (1986) predicted that the motion of a dislocation at sufficiently high $\epsilon$ should be captured by

$$v_d = \beta(\epsilon) (q - q_d(\epsilon)),$$

where $q_d$ is the wavenumber for which a dislocation does not move and $q$ is the background wavenumber (defined in Plapp et al. (1998)). It was argued (Cross & Tu, 1995; Cross, 1996; Li et al., 1996) that relation (11) should apply as well to the dynamics of the outer defects of giant spirals, i.e., $\omega_d = v_d(r)/r$ for $q = q(r)$. Plapp et al. (1998) verified this relationship both in experiments and in simulations. They found that for $\sigma = 1.4$ the wavenumber $q_d(\epsilon)$ decreased with increasing $\epsilon$. This is shown for the example of a cell of $\Gamma = 38.8$ in Fig. 17.

Away from the core of the spiral the wavefronts deviate only slightly from circular and their wavenumber-selection properties are thus well approximated by those of targets. Targets select a specific wavenumber $q(\epsilon)$ (Newell et al., 1990) (see also Sec. 6.2). If the prevailing $q$ differs from $q$, spirals will thus attempt to adjust their wavenumber by emitting circular waves of frequency $\omega_d$. From the nonlinear Cross-Newell phase-diffusion equation (Newell et al., 1990), one finds

$$\omega_d = \alpha(q_d(\epsilon) - q)/r;$$

where $\alpha = 2D(\parallel q), D(\parallel q)$ is the parallel diffusion constant, and $r$ is the distance from the center of the target. The numerical value of the parameter
$\alpha$ can be calculated from the growth rate $\sigma(qr)$ of the Eckhaus instability as

$$\alpha = (-2\sigma(qr)/K^2)_{k=0},$$

where $K$ is the wavenumber of the disturbance (Plapp et al. 1998).

Plapp et al (1998) measured $qr$ from the target patterns they observed for small $\epsilon$ and found quantitative agreement with the theoretical predictions by Buell & Catton (1986b). By extrapolating $qr(\epsilon)$ to larger $\epsilon$, and by measuring the spiral-selected wavenumber $q(\epsilon)$, they were able to determine $\alpha(\epsilon)$. Again they found good agreement with the numerically determined value obtained from the growth rate of the Eckhaus instability.

In the theoretical calculations it is implicitly assumed that mean-flow effects do not play an important role, at least not for the determination of the rotation frequency. In Fig. 18 a snapshot of the central region of a right-handed spiral is shown. Note that the induced mean flow is indeed confined to the core region. Surprisingly, the mean flow is in a direction which in principle has the tendency to unwind the spiral locally by advection. It seems clear that the coupling of the tips of multi-armed spirals by mean flow is responsible for the fast dynamics at their core.

In summary, the rotation of on-center, giant spirals can be understood in terms of two competing wavelength-selection mechanisms. As shown in Fig. 17, the revolving defects attempt to select a small wavenumber, while the target-like curvature of the rolls attempts to select a larger wavenumber. The rotating $n$-armed spirals select an intermediate wavenumber where both selection mechanisms are in balance. It has to be noted, however, that the more complex situations like the dynamics of off-center spirals, the core rotation, and the spiral instability are not understood and remain as a challenge for the future.

### 6.4 Textures

Here we discuss in more detail typical sidewall-mediated phenomena which occur in circular cells and which lead to increasing disorder and to persistent dynamics as $\epsilon$ is increased. For $\epsilon \geq 0.1$ we saw in Sect. 6 and Fig. 11 that the selection process
Figure 19: Patterns in circular cells for $\sigma \approx 1.0$. (a): $\epsilon = 0.09, \Gamma = 40$. (b): $\epsilon = 0.12, \Gamma = 41$. (c): $\epsilon = 0.34, \Gamma = 41$. From Hu et al (1993) and Hu et al (1995a).

due to cross rolls yields mean wavenumbers which, with increasing $\epsilon$, approach the SV instability. In addition, roll curvature increases and the obliqueness of the roll axes relative to the sidewalls decreases with increasing $\epsilon$. This process leads to a broadening wavenumber distribution throughout the cell, with the largest wavenumbers typically in the center of the sample. It is illustrated by images (a) and (b) in Fig. 19, and described quantitatively by the results shown in Fig. 20 which were obtained from real-space image-analysis by Hu et al (1995a). Figure 20a gives results for the average obliqueness $\beta \equiv \langle |n \mathbf{s}| \rangle$ ($\mathbf{n}$ is the normalized roll wavevector near the wall and $\mathbf{s}$ is the sidewall normal vector), where the average is taken along the sidewall and over many statistically independent images. One sees that $\beta$ decreases with increasing $\epsilon$ in the range $\epsilon \lesssim 0.17$ (for larger $\epsilon$ it remains essentially constant). The average roll curvature $\gamma \equiv \langle |\nabla \cdot \mathbf{n}| \rangle$, taken over the cell interior and over many statistically independent images, increases over the range $\epsilon \lesssim 0.13$, and then remains essentially constant until the onset of SDC near $\epsilon = 0.6$ causes it to rise dramatically.

As noted by Croquette (1989b), the compression of the rolls in the interior, which accompanies the enhanced roll curvature, causes the wavenumber in the cell center to exceed the SV instability when $\epsilon \approx 0.11$ and leads to the formation of dislocation pairs. This can be seen in Fig. 19b. These defects then travel toward the cell wall by a combination of climb and glide in a direction relative to the roll axes which is determined by the direction of the SV perturbation with the maximum growth rate (Hu et al, 1997). The result of this process is a reduction of $q$ to a value less than $q_{SV}$ and thus of a re-stabilization of the pattern. However, the domain walls emit new rolls which gradually re-compress the ones in the center and thus lead to a persistent time dependence. This case is an example of the creation of a time dependent state via the selection of an unstable state. A similar situation prevails for Taylor-vortex flow (TVF), where an Eckhaus-unstable state can be selected for a certain family of spatial ramps in the control parameter (Riecke & Paap, 1987; Ning et al, 1990; Wiener et al, 1997). The TVF case was examined in detail theoretically (Riecke & Paap, 1987). The selected wavenumber, the value of $\epsilon$ for the onset of time dependence, and the frequency of the events beyond onset were calculated, and there is good agreement between experiment (Ning et al, 1990) and theory. In the RBC case we do not know of predictions for the selection involving the domain walls, and
Figure 20: (a) sidewall obliqueness and (b) average roll curvature in a circular cell with \( \Gamma = 40 \) and \( \sigma = 1.0 \). The vertical bars give the standard deviations of the distributions used to calculate the average values. From Hu et al (1995a).

The problem involving curved rolls is two-dimensional and thus more complicated than the TVF case. Above the onset of time dependence the experiments showed that the process can be periodic or chaotic, apparently depending sensitively on \( \epsilon, \Gamma \), and \( \sigma \).

As \( \epsilon \) increases further, the domain walls shrink in length. Near \( \epsilon = 0.13 \) they typically have contracted to point singularities known as wall foci (Hu et al, 1994, 1995a). Examples can be seen in Figs. 19c and 10A. Meanwhile, the roll curvature has saturated at a value which is independent of \( \epsilon \) over the range \( 0.13 < \epsilon < 0.6 \) (see Fig. 20b). Like the domain walls, the wall foci emit rolls and thus keep the dynamics of the pattern sustained. The frequency of roll emission increases strongly with increasing \( \epsilon \) (Hu et al, 1995a). Other defect structures besides the SV-generated dislocations occur as well in this \( \epsilon \)-range, as can be seen in Fig. 19c. The most dominant are oscillating domain-wall structures in the cell interior where rolls meet at an angle (Hu et al, 1995a). Examples are visible near the top and bottom of Fig. 19c. The role of these various structures in the dynamics of the pattern is not understood in detail.

The evolution with \( \epsilon \) of the mean wavenumber\(^{17} \) \( < q > \) is shown as crosses in Fig. 21 for a sample with \( \Gamma = 40 \). As we saw in Sect. 5, \( < q > \) first moves to larger values as \( \epsilon \) increases. When SV events are first encountered near \( \epsilon \approx 0.11 \), \( < q > \) turns around and evolves along a line which is parallel to the SV boundary. Throughout this \( \epsilon \)-range, the wavenumber in the cell center is very close to \( q_{SV} \), and repetitive SV events are taking place. Near \( \epsilon = 0.6 \), where SDC first emerges, \( < q > \) moves further away from the SV boundary and into the Busse-balloon interior.

Interestingly, the behavior of \( < q > \) is quite different for very large \( \Gamma \). This is illustrated by the data taken by Morris et al (1996), which are for \( \Gamma = 78 \) and

\(^{17} < q > \) was determined by calculating the first moment of the structure function \( S(q) \) (the azimuthal average of the square of the modulus of the Fourier transform of the images, see Morris et al (1996)).
Figure 21: Selected mean wavenumbers as a function of $\epsilon$. The solid circles are from Morris et al (1996) for $\Gamma = 78$ and $\sigma = 0.95$. The crosses are from Hu et al (1995a) for $\Gamma = 40$ and $\sigma = 0.98$. The large open circles are the onset at $\epsilon_s$ of SDC. After Hu et al (1995a).

which are given in Fig. 21 by the solid circles. In this case the system never approaches the SV instability. Apparently, in very large systems different types of defects appear in the pattern even at quite small $\epsilon$ and lead to a selection mechanism which differs from the domain-wall mechanism. The different path in the $\epsilon - q$ plane leads to the onset of SDC at the much smaller value $\epsilon_s \approx 0.2$.

6.5 Hexagons.

Experiments in RBC are usually designed to yield small temperature differences, so that the temperature dependence of material properties may be neglected and the Oberbeck-Boussinesq approximation (see Sect. 3) may be used in the corresponding theory. In that case the bifurcation is supercritical and leads to ISR as discussed in Sects. 3 and 5. However, in experiments with larger temperature differences non-Oberbeck-Boussinesq (NOB) effects are important. They have been discussed theoretically by several authors, and most systematically by Busse (1967) who introduced the parameter

$$\mathcal{P} = \sum_{i=0}^{4} \gamma_i \mathcal{P}_i$$

to describe them quantitatively. Here $\gamma_0 = -\Delta \rho / \rho$, $\gamma_1 = \Delta \alpha / \alpha$, $\gamma_2 = \Delta \nu / \nu$, $\gamma_3 = \Delta \lambda / \lambda$, and $\gamma_4 = \Delta C_p / C_p$. The quantities $\Delta \rho$, etc. are the differences in the values of the property at the bottom (hot) and top (cold) end of the cell, and $\rho$, etc. are their average values. The coefficients $\mathcal{P}_i$ were calculated to leading order by Busse (1967). A new calculation (Tschammer, 1997) to $O(1/\sigma)$ yielded\(^{18}\)

$$\mathcal{P}_0 = 2.676 - 0.361/\sigma,$$

$$\mathcal{P}_1 = -0.631 - 0.772/\sigma,$$

$$\mathcal{P}_2 = 2.765,$$

\(^{18}\)The value of $\mathcal{P}_3$ differs significantly from the original calculation by Busse (1967).
\[ p_3 = 9.54, \]
\[ p_4 = -6.225 + 3857/\sigma. \]

In NOB convection the initial bifurcation is transcritical, and the nonlinear state which forms beyond it consists of hexagonal cells. Consistent with the vertical variation of the fluid properties, hexagons break the mirror symmetry at the horizontal midplane of the sample because upflow and downflow at their centers are not equivalent. For positive \( \mathcal{P} \) (gases) hexagons have downflow in their centers, while for negative \( \mathcal{P} \) (liquids) that flow is upward (Graham, 1993; Buse, 1989). To a good approximation the hexagonal patterns near onset can be described by three coupled real Ginzburg-Landau equations which have a potential. Thus many aspects of pattern formation in this system can be understood in variational terms.

![Hexagonal patterns](image)

Figure 22: Transcritical bifurcation to hexagons \( (\Gamma = 86, \sigma = 1) \). (A): Fluctuating pattern below the onset of convection. (B): Nucleation of hexagons and (C): the same \( \epsilon \) as (B) but after transients have died out. (D) – (F): \( \epsilon \) is quasistatically decreased. The circular pattern in the upper left hand of the figures is caused by a dust particle. The spot at the lower right was caused by reduced reflectivity of the bottom plate. Neither inhomogeneity seemed to influence the experimental observations. After Bodenschatz et al. (1991).

NOB convection was investigated in small-aspect-ratio systems and interesting results for the roll-to-hexagon transition were obtained (Ciliberto et al, 1988; Perez-Garcia et al, 1990; Pampaloni et al, 1992). Penta-hepta defects in the hexagonal pattern were investigated both experimentally (Ciliberto et al, 1990, 1991) and theoretically (Pismen & Nepomnyashchy, 1993; Rabinovich & Tsimring, 1994; Tsimring, 1995, 1996). RBC with compressed gases made possible larger-aspect-ratio experiments with much higher resolution in \( \epsilon \) than could be done before (Bodenschatz et al, 1991, 1992a,b, 1993). Bodenschatz et al (1991)
Figure 23: (A) A defect-free hexagonal pattern. (B–D) Rolls nucleate at the sidewalls and propagate into the pattern. (B–C–D) are spaced $1500\epsilon_0$ apart. After Bodenschatz et al (1991, 1992a).
Figure 24: Target (A), one-armed spiral (B), and two-armed spiral (C). (D): Pattern during the transient from rolls to hexagons $2000 \epsilon_0$ after $\epsilon$ was decreased from the stable target (A) at $\epsilon = 0.108$ to $\epsilon = 0.103$. After Bodenschatz et al (1991, 1992a).
were able to resolve the theoretically predicted hysteresis associated with the transcritical bifurcation from conduction to hexagons (Busse, 1967). This is shown in Fig. 22. As $\epsilon$ was increased quasistatically, first only fluctuations (see Sect. 4) occurred (Fig. 22A), and then hexagons were nucleated and expanded to fill part of the cell (Fig. 22B, C). This first value of $\Delta T_c$ at which the pattern formed was taken as $\Delta T_c^*$, corresponding to $\epsilon = 0$ at that spatial location. The true value of $\Delta T_c$ for the deterministic system should actually be slightly larger because for a transcritical bifurcation the fluctuations in the experiment would be expected to cause an early transition. The limited size of the patch in Fig. 22C can be attributed to very small variations of the cell thickness which cause a slight spatial variation of $\epsilon$. Clearly visible in Fig. 22 are the fronts that separate regions with and without convection. In steady state the fronts are located at those spatial positions where the local $\epsilon$-value is equal to $\epsilon_f$, where the potentials of the convecting and the conducting states are equal. When $\epsilon$ was decreased quasistatically, the hexagons shrunk to a smaller patch (Fig. 22D, E) and disappeared when $\epsilon$ reached $\epsilon_f$ at their location (Fig. 22F). For the GL equations one can show that $\epsilon_f = \frac{4}{3} \epsilon_a$. Thus the measurement $\epsilon_f = -1.93 \times 10^{-3}$ corresponds to $\epsilon_a = -2.17 \times 10^{-3}$. This value may actually be slightly too small because of the influence of the fluctuations on the measured $\Delta T_c$. The size of the hysteresis loop $-\epsilon_a$ of the deterministic transcritical bifurcation is given by the theory of Busse (1967) \textsuperscript{19} as $\epsilon_a = -1.6 \times 10^{-3}$, somewhat smaller than the experimental value. The reason for this difference is not known.

For $\epsilon = 0.02$, perfect hexagonal patterns like the one shown in Fig. 23A were found in the experiment by Bodenschatz et al. (1991). Remarkably, even when $\epsilon$ was increased by a jump, initial grain boundaries and defects annihilated and a well-ordered pattern was reached after about 15$\tau_b$. With further quasistatic increase of $\epsilon$, rolls nucleated at the sidewalls above a certain $\epsilon$-value. After a long transient a pattern consisting of a single N-armed, rotating giant spiral was formed. Discrete steps in $\epsilon$ of different sizes from the hexagon to the roll regime were used to produce spirals with $0 < n \leq 13$ (Bodenschatz et al., 1992a). The initial evolution of the nucleation of rolls is shown in Figs. 23B, C, and D. When $\epsilon$ was quasistatically decreased from the n-armed spiral state, spirals like the one shown in Fig. 24A was formed. With further decrease in $\epsilon$, hexagons nucleated at the sidewalls as shown in Fig. 24B, and C decreased the number of their arms in steps of one until a target pattern (Fig. 24A) was formed. With further decrease in $\epsilon$, hexagons nucleated at the sidewalls as shown in Fig. 24D. After a transient a perfectly ordered hexagonal pattern was formed once again. As was the case near $\epsilon_f$, the transitions between hexagons and rolls could also be understood in terms of the potential of the relevant Ginzburg-Landau equations. This transition occurred near $\epsilon_T$ where rolls and hexagons had the same potential.

Another interesting topic is the structure and dynamics of penta-hepta defects in hexagonal patterns, where instead of two neighboring hexagonal lattice cells a pair of a pentagonal and a heptagonal cells is observed.

\textsuperscript{19} The estimate of the theoretical value of $\epsilon_a$ by Bodenschatz et al. (1991) has to be reconsidered. The pressure was printed incorrectly as 21.8 bar, but the correct value 23.1 bar was used in the analysis. The equation of state for CO$_2$ is now better known (de Bruyn et al., 1996), and the new values of $P_f$ given above should be used. One obtains $P_f = 3.3$ for this experiment, which yields $\epsilon_a = -1.6 \times 10^{-3}$, coincidentally quite close to the value cited in the original paper. Thus the conclusions of Bodenschatz et al. (1991) are unchanged.
dislocation), while the third one remains finite. An example of a moving penta-hepta defect is shown in Fig. 25A. Also shown in Fig. 25C and D is the Fourier demodulation into the three sets of rolls that make up the hexagonal pattern of Fig. 25A at two different times separated by 70.6t₀. In this example the penta-hepta defect moved with a combination of glide and climb of the dislocations in the two roll patterns, while the third roll pattern remained dislocation free. The dynamics of penta-hepta defects has been investigated theoretically (Pismen & Nepomnyashchy, 1993; Rabinovich & Tsimring, 1994; Tsimring, 1995, 1996). Based on the three coupled Ginzburg–Landau equations (Busse, 1967), Tsimring (1995, 1996) calculated the mobility of an isolated penta-hepta defect. He also considered the interaction of a pair of penta-hepta defects. Only future experiments will be able to tell whether the predictions of this theory are as successful as their analogue in the case of electro-convection of nematic liquid crystals (Kramer et al, 1990).

Recently, in an experiment using compressed SF₆ gas, Assenheimer & Steinberg (1996) observed the coexistence of the two types of hexagons with upflow and downflow at their centers. Later Bajaj et al (1997) confirmed this observation in an experiment using acetone as a fluid. A picture of such a convection state
Figure 26: Coexisting up- and down-flow hexagons together with rolls for $\Gamma = 80$ and $\sigma = 4.5$. The hexagon wavelength is about 20% larger than the roll wavelength. From Assenheimer & Steinberg (1996).

is shown in Fig. 26. Assenheimer & Steinberg (1996) found such patterns in a cylindrical cell of aspect-ratio $\Gamma = 80$ for a Boussinesq fluid with $P < 0.09$ and $2.8 \leq \sigma \leq 28$ at $\epsilon \approx 4$. They observed that the hexagonal planforms nucleated at the centers of targets and spirals when $\epsilon$ was increased quasistatically (Assenheimer & Steinberg, 1996; Aranson et al, 1997). This is shown in a sequence of images in Fig. 27.

Figure 27: Experimental hexagon nucleation at a spiral core for $\sigma = 4.5$ and $\epsilon = 3.19$. The pictures are spaced $3.6t_v, 3.6I_v, 22.7t_v, 18.0t_v,$ and $10.7t_v$ apart. From Assenheimer & Steinberg (1996).

Bajaj et al (1997) observed the patterns for a smaller circular cell of $\Gamma = 55$ for $P < 0.4, \sigma = 4$, and $\epsilon > 4.5$. In their experiment the hexagons nucleated near sidewall foci. Clever & Buse (1996) showed numerically that up-down hexagons constitute another stable attractor aside from rolls for the experimental values of $\sigma$. 
7 SPIRAL–DEFECT CHAOS

For $\sigma \approx 1$ Morris et al (1993) discovered a spatiotemporally chaotic state in a circular large-aspect-ratio cell with $\Gamma = 78$ while increasing $\varepsilon$ above 0.26. This novel state is now referred to as spiral-defect chaos (SDC). A snapshot is shown in Fig. 28. SDC is characterized by complex spatio-temporal dynamics involving rotating spirals, targets, dislocations, disclinations, and grain boundaries. The spirals can be right-handed or left-handed, and single-armed or multi-armed, and thus have common features with the giant spirals discussed in Sect. 6.3. This new state is a nice example for the little-understood phenomenon of spatio-temporal chaos (see, for instance, Gol'ub (1994)).

Figure 28: A snapshot of SDC in a circular cell of $\Gamma = 78$ at $\varepsilon = 0.72$ and $\sigma = 0.96$. From Morris et al (1993).

Figure 29: (A): Initial shadowgraph picture of SDC for $\varepsilon = 0.7$ and $\sigma = 1$. (B): Shadowgraph picture and (C): numerically simulated mid-plane temperature-field after 20$t_c$ (only the central section of the sample is shown). After Decker (1995).

Much recent experimental (see, for instance, Ahlers, 1998, and references...
therein) and theoretical (see, for instance, Cross, 1996, and references therein) work has provided convincing evidence that SDC is an intrinsic state of RBC for fluids with $\sigma \approx 1$. In distinction to the features of the textured patterns discussed in Sec. 6.4, SDC is not caused or substantially influenced by the sidewalls. It occurs with rectangular (Morris et al., 1996) as well as with circular (Morris et al., 1993) sidewalls, provided the aspect ratio is not too small. SDC was found numerically in solutions of a generalized Swift-Hohenberg equation (Xi et al., 1993; Bestehorn et al., 1992) and of the Boussinesq equations (Decker et al., 1994; Pesch, 1996) (see Sect. 3).

In their simulations Decker et al. (1994) were able to use an experimental shadowgraph picture from Morris et al. (1993) as initial conditions (see Sect. 3) and reproduce the experimentally observed dynamics at least over a modest time interval. This is illustrated in Fig. 20. The numerical dynamics resembles the experiment quite closely. The small differences may easily be explained by the difference in boundary conditions and the sensitivity of ("chaotic") SDC to small differences in initial conditions. The code can thus be used to generate reliably the full three-dimensional temperature and velocity fields which are not accessible with the experimental shadowgraph-technique. Of particular interest is the opportunity to determine the mean-flow field (or vertical vorticity), which is important for the dynamics of SDC (Xi et al., 1993; Decker et al., 1994; Cross, 1996) but difficult to determine experimentally, by using the experimental image as the initial condition for a relatively brief numerical integration. Figure 30 shows the strength of the vertical vorticity overlaid onto the convection pattern. The vertical vorticity is clearly localized at high curvature regions, i.e., at spiral cores, grain boundaries, and dislocations. Apart from numerical simulations, we know of only one direct experimental visualization of the mean flow. This was done for an off-center target pattern by Croquette et al. (1986b) by using a photochromic technique in an experiment with water.

![Figure 30: Vertical vorticity field overlaid on top of the convection pattern for $\epsilon = 0.75$ and $\sigma = 1.1$. Regions of high vorticity appear bright. From simulations by IV Melnikov and W Pesch, unpublished.](image)

In experiments using compressed $SF_6$ gas, Assenheimer & Steinberg (1993,
1994) found a transition from SDC to target chaos when \( \sigma \) was increased above approximately 4 by tuning the temperature and pressure near the critical point of the gas. This is consistent with the notion that for larger Prandtl numbers mean-flow effects are weaker. Such transitions have been confirmed in numerical studies (Decker, 1995; Cross, 1996), but are not understood in detail.

![Figure 31: The onset \( \epsilon_s \) of SDC as a function of aspect ratio \( \Gamma \) and Prandtl number \( \sigma \) for circular cells (except where noted). (a): \( \sigma \approx 1 \); Solid circles: \( \Gamma = 29, 70 \) and 100 from J Liu, KMS Bajaj, and G Ahlers (unpublished); open circle: \( \Gamma = 40 \) from Ecke & Hu (1997); triangle: \( \Gamma = 52 \) and 60 from Hu et al (1995c); square: \( \Gamma = 75 \) from Morris et al (1996); upside down triangle: \( \gamma = 50 \) for a square cell from Cakmur et al (1997a). The solid line is a guide to the eye. (b): \( \Gamma = 30 \) (triangles) and \( \Gamma = 70 \) (circles). The open circles are for pure gases, and the solid ones are for gas mixtures. After Liu & Ahlers (1996).](image)

7.1 Onset

The onset value \( \epsilon_s \) of SDC decreases as \( \Gamma \) increases. This is shown in Fig. 31a. The influence of cell geometry on \( \epsilon_s \) appears to be weak. For example, for \( \sigma \approx 1 \) Hu et al (1995c) and Cakmur et al (1997a) found nearly the same value \( \epsilon_s \) for a circular and a square cell with the same \( \Gamma \approx 50 \). There seem to be two regimes with different \( \epsilon_s \). For \( \Gamma \leq 50 \) the onset occurs near \( \epsilon_s \approx 0.6 \), whereas for \( \Gamma \geq 70 \) SDC is found already for \( \epsilon > 0.2 \). As can be seen in the figure, the transition from one regime to the other occurs over the relatively small \( \Gamma \)-range from 50 to 70. There is no indication that \( \epsilon_s \) approaches zero as \( \Gamma \) becomes large, as had been suggested by Li et al (1998). The SDC onset also has an interesting dependence on \( \sigma \). Using mixtures of gases, it was possible to reach \( \sigma \)-values as small as 0.17 (Liu & Ahlers, 1996). Results for two values of \( \Gamma \) are shown in Fig. 31b. The data suggest that \( \epsilon_s \) remains finite with a value close to 0.1 as \( \sigma \) vanishes.

7.2 Spatio-temporal evolution

For circular cells of \( \Gamma = 40 \) and 78, the mean wavenumbers \( < q > \) obtained by Fourier analysis of images are shown in Fig. 21. Since, at small \( \epsilon \), large- and small-aspect-ratio systems are subject to different wavenumber-selection mechanisms
(see Sect. 6.4), the SDC onset is also different. The results suggest that there is a separatrix (roughly a straight line through the two large circles in Fig. 21) below and to the right of which SDC does not occur. As SDC evolves above that line, it provides its own selection mechanism. Thus the data sets in Fig. 21 for the two aspect ratios approach each other, and for $\epsilon \geq 0.6$ select the same $<q>$ in fully developed SDC. The shape of the convection cell is apparently of minor importance since (at least for $\epsilon \leq 1$) almost the same $<q>$ is selected in equi-aspect-ratio cells with circular (Hu et al., 1995c) and square (Gakmur et al., 1997a) geometry.

Time series of shadowgraph images of SDC were analyzed by constructing the three-dimensional structure factor $S(k, \omega)$ of the shadowgraph intensity (Morris et al., 1996). These data were used to characterize the $\epsilon$-dependence of the translational correlation length and the correlation time of the chaotic state (Hu et al., 1995c; Morris et al., 1996). Both were found to decrease approximately as a power law in $\epsilon$, but a theoretical explanation of this behavior remains to be developed.

![Figure 32: The probability $p(m)$ of finding $m$ spirals in a given image for $\epsilon = 0.96$. The solid line is a Poisson distribution. After Ecke et al. (1995).](image)

An interesting aspect of SDC is the statistics of its time dependence. The number of spirals present at any instance fluctuates. Figure 32 gives the experimentally determined probability $p(m)$ of finding $m$ spirals in a given snapshot of the pattern (Ecke et al., 1995). It is fit well by a Poisson distribution (solid line). It should be noted that there are no adjustable parameters, since the Poisson distribution is determined completely by the average number of spirals $<m>$ which in turn is determined separately from the same set of images. The agreement between the data and the Poisson distribution function is quite good, implying that the birth and death of a particular spiral is not significantly dependent on the presence of the others, i.e., that the spirals to a good approximation may be regarded as “non-interacting excitations” of the system (see Landau & Lifshitz, 1958).

7.2.1 Coexistence with ideal straight rolls

We saw already in Sect. 5 that SDC is found in a parameter region where (besides targets and giant spirals) ideal straight rolls (ISR) are stable as well. The Buse Ballon, giving the stability range of ISR, was shown in Fig. 6. That figure
included also the wavenumbers corresponding to the maxima of the azimuthal averages of the moduli of the Fourier transforms of SDC images, and these were located deep in the interior of the balloon. Similar data for the mean wavenumber of SDC are found in Fig. 21. Images which compare SDC and ISR in the same cells and at the same $\sigma$ and $\epsilon$ are shown in Fig. 5. Additional confirmation of the bistability comes from integrations of the Boussinesq equations with periodic boundary conditions which always yield SDC from random initial conditions but give stable ISR when ISR-like initial conditions are used (Decker et al., 1994). In the experiments SDC is the generically selected state above $\epsilon_s$ when $\epsilon$ is quasistatically increased from onset (see Liu & Ahlers, 1996, and references therein). One concludes that the boundary-induced disorder (foci, dislocations, and grain boundaries) which evolves as $\epsilon$ is increased place the system into the SDC attractor basin before $\epsilon_s$ is reached. Special initial conditions are needed to reach the competing attractors, as discussed in Sect. 5.

Figure 33: Orientational correlation lengths $\xi_o$ compared to $\xi_s$ (see text) for $\Gamma = 50, \sigma = 1.03$. From Egolf et al (1998).

Several quantities have been used to characterize the onset of SDC at $\epsilon_s$. For circular cells Hu et al (1995a,c) demonstrated that the global curvature of the patterns (see Fig. 20b) suddenly increased at $\epsilon_s$. Another method consisted of counting the number $m$ of spirals present in many images of a given finite sample and determining where their averages $<m>$ vanish as a function of $\epsilon$ (Ecke et al., 1995; Liu & Ahlers, 1996; Ecke & Hu, 1997). The question was addressed also for square cells (Cakmur et al., 1997a; Egolf et al., 1998). When $\epsilon_s$ was approached from above, the spatio-temporal evolution of SDC was dominated by progressively larger regions of ISR and larger spirals. This was quantified by Egolf et al (1998). Figure 33 shows the experimentally observed rapid growth of the orientational correlation length $\xi_o$. Egolf et al (1998) showed that the transition between ISR and SDC occurred as $\xi_o$ approached the system size in their experiment. They also found $\xi_o$ to be proportional to a characteristic length $\xi_s$ which is a measure of the density $n_d$ of targets and spirals. One finds $\xi_o \approx \xi_s = \sqrt{n_d/\pi}$ (Fig. 33). In addition the pattern appeared to fluctuate intermittently between a disordered state and almost perfect ISR. This is illustrated in Fig. 34. Cakmur et al (1997a) quantified this behavior by using the spectral pattern entropy $S(t) = -\sum_k p(k,t) \ln p(k,t)$ (Neufeld & Friedrich, 1994; Xi & Gunton, 1995).

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20J Liu, KMS Bajaj, and G Ahlers (unpublished) recently found similar behavior for a circular cell of $\Gamma = 109$. That work was presented at the workshop Spatiotemporal Characterization of Spiral Defect Chaos at Los Alamos National Laboratory, January 4-5 1999.
Figure 34: Time evolution of the pattern entropy $S$ and the pattern for $\epsilon = 0.554$: (a) $14.52t_H$, (b) $20.27t_H$, (c) $30.95t_H$, (d) $32.32t_H$. From Cakmur et al (1997a).

where $p(k, t)$ is the spectral distribution function which describes the normalized power in mode $k$ at time $t$. $S(t)$ measures the disorder in the pattern. For example, if the pattern is ideal, i.e., when only a pair of conjugate modes is excited, $S = \ln 2$ and otherwise $S > \ln 2$. In Fig. 34 the time evolution of the pattern entropy is shown for $\epsilon = 0.554$. The degree of spatial order of the patterns as judged by eye shows a clear correlation with the value of the pattern entropy. Cakmur et al (1997a) observed that the well ordered patterns during the evolution were typically aligned either diagonal or perpendicular to one of the sidewalls of the square cell; it appeared that the pattern was probing the system's symmetries. In Fig. 35(A) the temporal average of the pattern entropy $\langle S \rangle_t$ as a function of $\epsilon$ is shown. As the transition to ISR is approached from above, $\langle S \rangle_t$ shows a sharp decrease. Fig. 35(B) shows the standard deviation $\sigma(S)$ over the

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21 If only half of the modes in Fourier space are taken an ideal pattern would have $S = 0$. 
Figure 35: (A) The time averaged pattern entropy \( \langle S \rangle_t \) and (B) the standard deviation \( \sigma(S) \) as a function of \( \epsilon \). After Cakmur et al. (1997a).

same range of \( \epsilon^{\text{22}} \). As \( \epsilon \) is approached, \( \sigma(S) \) displays a sharp increase. It is also possible to measure \( \sigma(S) \) below \( \epsilon(S) \) in the fluctuating region of SDC before a strong fluctuation leads to ISR. This is also shown in Fig. 35B. For \( \epsilon \gg \epsilon_a \) the standard deviation \( \sigma(S) \) approaches a small value, suggesting that the system consists of many independent, fluctuating subsystems. The behavior shown in Fig. 35 appears to be similar to a second order equilibrium phase transition in a finite size system, where \( \langle S \rangle_t \) would correspond to the internal energy and \( \sigma(S) \) to the specific heat. On the other hand, the well established bistability of SDC and ISR has similarities with a first-order equilibrium phase transition. It is not clear at this point whether the comparison to equilibrium phase transitions is justified or even helpful. Other approaches to understand SDC as a competition between two attractors might be in terms of blowout bifurcations (Ashwin, 1998). However, an extension of this theory to spatio-temporal chaotic systems so far has not been developed.

In another experiment the spatio-temporal dynamics of SDC was analyzed by a Karhunen-Loeve decomposition (Zedli et al., 1998). This analysis suggests that the spatio-temporal chaotic properties of SDC are extensive. This is consistent with the \( \epsilon \)-independence of \( \sigma(S) \) for sufficiently large values of \( \epsilon \) (Fig. 35B).

When initializing an almost perfect ISR state, Cakmur et al. (1997a) observed that SDC can propagate into ISR. This was investigated in more detail for a rectangular cell of aspect ratio 100 \( \times \) 50 by Jeanjean (1997) where an ISR state was initialized with the same method as that used by Cakmur et al. (1997a). For a fixed value of \( \epsilon \), when the experiment was started with a perfect stripe pattern, SDC nucleated after long times only in one corner of the convection cell. Jeanjean (1997) attributed this to a slight geometrical inhomogeneity. Then SDC grew at the sidewalls by compressing the ISR. After an initial transient, where ISR adjusted their wavenumber towards the SV-instability boundary, SDC started propagating into the ISR state with a flat front. An example of the evolution of SDC propagating into ISR is shown in Fig. 36. Local chaotic fluctuations compressed the rolls at the boundary between SDC/ISR, hereby increasing the local wavenumber above the SV-instability boundary. A SV-instability occurred and as described in Sect. 5 a dislocation pair nucleated which moved towards other dislocations or the boundaries so as to decrease the number of roll pairs.

\(^{22}\)The quantity \( \sigma(S) \) given by Cakmur et al. (1997a) is the standard deviation of \( S \) from its mean value, rather than the variance as stated by the authors.
Figure 36: SDC propagates into ISR with a flat front for $\epsilon = 0.71$. The figures are spaced $\Delta t = 5002\tau_0$ apart. In (B) the front is sufficiently close to the opposite sidewall to trigger nucleation of defects. From Jeanjean (1997).

This process continued as SDC gradually replaced the straight rolls. It was found that the front propagated with constant velocity whose magnitude increased with increasing $\epsilon$, as shown in Fig. 37. Below $\epsilon_3$ Jeanjean (1997) found a drifting pattern which appeared to be dominated by wall foci. It was suggested that the front speed may be limited by the spatiotemporal dynamics that lead to SV-instabilities (IV Melnikov, DA Egolf, E Bodenschatz, unpublished). However, the understanding of this very interesting property of RBC remains a challenge for the future.

One of the puzzling features of SDC was that its average wavenumbers as determined by Fourier transforms (Morris et al., 1993) or by local measurements in physical space (Egolf et al., 1998; Bowman & Newell, 1998) were in the middle of the stability region for ISR. A small tail of the local wavenumber distribution was found to be above the SV-instability boundary, while only an unnoticeable amount was below the CR-instability boundary (Egolf et al., 1998). The tail at large $q$ corresponds to small compressed roll patches with a local wavevector in the SV-unstable regime of the Busse balloon. An example of the wavenumber distribution is shown in Fig. 38, which also compares the local wavenumber with the azimuthally averaged powerspectrum. It is an important experimental observation that the stability boundaries for infinite systems (Busse balloon) apply locally in a disordered SDC pattern (Egolf et al., 1998). By the well known pinching mechanism (see Sect. 6.4) dislocations are nucleated in the compressed regions.
8 ROTATION ABOUT A VERTICAL AXIS

Rotation inhibits the onset of convection and increases the critical wavenumber of the pattern at onset (Chandrasekhar, 1961). Figure 30 shows experimental and theoretical results for $R_c(\Omega)$ and $q_c(\Omega)$. We see that, at the linear level, there is excellent agreement between experiment and theory. Both $R_c(\Omega)$ and $q_c(\Omega)$ are predicted to be independent of $\sigma$.

As we saw in Sec. 5, RBC sufficiently close to onset ($\epsilon \lesssim 0.1$) is relatively simple and, in the absence of boundary forcing, consists of time independent straight rolls. The system becomes much more complex and interesting even near onset when it is rotated about a vertical axis with an angular velocity $\dot{\Omega} \equiv (d^2/\nu)\Omega \hat{z}$ ($\dot{\Omega}$ is the rotation rate in radian per second and $\hat{z}$ is the unit vector in the vertical direction). In that case the Coriolis force proportional to $\hat{\Omega} \times \vec{v}$ acts on the convecting fluid (here $\vec{v}$ is the fluid velocity field in the rotating frame) and renders the system non-variational. Thus time dependent states can occur arbitrarily close to onset. Over a wide parameter range the bifurcation remains supercritical for $\Omega > 0$, i.e. the flow amplitudes still grow continuously from zero and the usual weakly-nonlinear theories, for instance in the form of Ginzburg-
Landau (GL) or SH-equations, should remain applicable. Thus one may expect interesting new effects to occur in a theoretically tractable parameter range.

8.1 Small rotation rates

In the range $\Omega \lesssim 10$ a number of interesting, albeit somewhat complicated and as yet incompletely understood, phenomena occur. Although at onset the pattern seems to consist of time independent rolls, for small $\epsilon$ these rolls become curved and assume an S-shape. As $\epsilon$ increases slightly, formation of defects adjacent to the sidewall, gliding (Millan-Rodriguez et al, 1995) and climbing of defects through the cell interior, and the motion of walls between domains of different roll orientation becomes prevalent. A significant fraction of this dynamics seems to be induced by the sidewalls. We will not discuss these interesting phenomena in detail, and instead refer the reader to the papers by (Hu et al, 1997, 1998) for recent results and for references to earlier literature.

One noteworthy aspect of moderate rotation rates and somewhat larger $\epsilon \gtrsim 0.5$ is the influence of the rotation on SDC (Ecke et al, 1995). It turns out that for $\Omega = 0$ the average number of righthanded ($r$) and lefthanded ($l$) spirals is equal. With rotation, this chiral symmetry is broken. A useful order parameter to describe this phenomenon is $M = (m_l - m_r)/(m_l + m_r)$ where $m_l$ and $m_r$ are the average number of lefthanded and righthanded spirals respectively. The parameter $M$ can vary from -1 to 1 and vanishes for $\Omega = 0$. Results for $M$ are shown in Fig. 40. It turns out that $M(\Omega)$ can be described well by a hyperbolic tangent or by a Langevin function, as shown by the solid line in the figure. This is reminiscent of the magnetization $M(H)$ of a dilute paramagnet as a function of the applied magnetic field $H$. 
Figure 40: The difference $M = (m_L - m_R)/(m_L + m_R)$ between the average number of left-handed ($m_L$) and right-handed ($m_R$) spirals, normalized by the total number ($m_L + m_R$), as a function of the rotation frequency $\Omega$. The solid line is the function $M = \tanh(\Omega/\Omega_0)$ with $\Omega_0$ adjusted to fit the data. After Ecke et al. (1995).

8.2 Domain (or K"{u}ppers-Lortz) chaos

For $\Omega > \Omega_c$, the primary bifurcation leads immediately to a state of spatio-temporal chaos in the form of rolls which are unstable (K"{u}ppers & Lortz, 1960; K"{u}ppers, 1970; Clever & Busse, 1979; Niemela & Donnelly, 1986). Although $\Omega_c$ depends on the Prandtl number $\sigma$, it has a value near 14 for the $\sigma$-values near unity which are characteristic of compressed gases. The instability is to plane-wave perturbations which are advanced relative to the rolls at an angle $\Theta_{KL}$ in the direction of $\Omega$. This phenomenon is known as the K"{u}ppers-Lortz instability. A snapshot of the resulting nonlinear state of convection is shown in Fig. 41 (Hu et al., 1995b). The pattern consists of domains of rolls which incessantly replace each other, primarily by irregular domain-wall motion (see e.g. Hu et al., 1998, and references therein). The spatial and temporal behavior suggests the term 'domain chaos' for this state. For $\sigma < 0.33$, the primary bifurcation is expected to be supercritical both below and above $\Omega_c$.

The opportunity to study STC at onset has led to renewed recent theoretical and experimental interest in the KL-state. For $\sigma \approx 1$ and $\Omega \approx 20$ it was demonstrated experimentally with high resolution (Hu et al., 1997) that the bifurcation is indeed supercritical, and that it leads to continuous domain switching through a mechanism of domain-wall propagation even at small $\epsilon$ (Bodenschutz et al., 1992a; Hu et al., 1995b, 1997, 1998) This qualitative feature has been reproduced by Tu & Cross (1992b) in numerical solutions of appropriate coupled GL-equations, as well as by Neufeld et al. (1993) and Cross et al. (1994) through numerical integration of a generalized SH-equation.

Of interest are the time and length scales of the KL instability near onset. The GL-model assumes implicitly a characteristic time dependence which varies as $\epsilon^{-1}$ and a correlation length which varies as $\epsilon^{-1/2}$. Measurements of a correlation length given by the inverse width of the square of the modulus of the Fourier

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23De Bryn et al. (1995); Hu et al. (1995b); Zhong et al. (1991); Zhong & Ecke (1992); Tu & Cross (1992b); Fantz et al. (1992); Neufeld et al. (1993); Cross et al. (1994); Ponty et al. (1997)
Figure 41: An example of the Klüppers-Lortz unstable rolls for $\Omega = 15.4$ and CO$_2$ at a pressure of 32 bar with $\sigma = 1.0$ and $\epsilon = 0.05$. After Hu et al (1995b).

transform as well as a domain-switching frequencies as revealed in Fourier space yielded the data in Fig. 42 (Hu et al, 1995b, 1997). These results seem to be inconsistent with GL-equations since they suggest that the time in the experiment scales approximately as $\epsilon^{-1/2}$ and that the two-point correlation length scales approximately as $\epsilon^{-1/4}$.

These results also differ from numerical results based on a generalized SH-equation (Cross et al, 1994) although the range of $\epsilon$ in the numerical work is rather limited. The disagreement between experiment and theory is a major problem in our understanding of STC. It is interesting to note that very recently Laveder et al (1999) were able to generate numerical solutions of a stochastic SH-model which yield an $\epsilon$ dependence of $\xi$ quite similar to that of the experimental results in Fig. 42. However, quite large noise intensities were required in the model, far in excess of any noise source in the experiment. A more likely candidate for an explanation was suggested by Hu et al (1998). As noted by Hu et al (1997) at relatively small $\Omega$, the sidewalls generate defects which travel into the sample interior. Since the defects travel at a constant speed rather than diffusively, they have a large range and possibly can influence the system interior even when $\Gamma$ is quite large as in the experiment. It is conjectured that the defects break up the KL-domains and thus alter the characteristic length and time scale. It remains to test this hypothesis experimentally as well as by numerical integrations of the Boussinesq or SH-equations with realistic lateral boundary conditions.

$^{24}$It was shown by Hu et al (1998) that the data for $\xi$ and $\omega_n$ can be fit reasonably well with a powerlaw and the expected theoretical leading exponents if large correction terms are allowed in the analysis, but even then there are significant systematic deviations at small $\epsilon$ particularly for $\xi$. 
8.3 Squares near onset

Motivated by the unexpected scaling of length and time with ε for the KL-state at Ω < 20, new investigations were undertaken recently in which the range of Ω was significantly extended to larger values. Contrary to theoretical predictions (Clever & Busse, 1979; Clune & Knobloch, 1993), it was found (Bajaj et al., 1998) that for Ω ≥ 70 the nature of the pattern near onset changed qualitatively although the bifurcation remained supercritical. Square patterns like the one shown in Fig. 43 were stable, instead of typical KL-patterns like the one in Fig. 41. The squares

Figure 43: Examples of square patterns near onset. (a): Argon gas with σ = 0.69 for ε = 0.01 and Ω = 145. (b): water with σ = 5.4 for ε = 0.09 and Ω = 170. (c): numerical integration of the Boussinesq equations (see Sect. 3.4) for σ = 5.3, ε = 0.06, and Ω = 60. (a) and (b): Bajaj et al. (1998). (c): O Brausch, W Pesch, unpublished.

occurred both when Argon with σ = 0.69 was used (Fig. 43a) and when the fluid was water with σ ≈ 5 (Fig. 43b). Subsequently they were found also in numerical integrations of the Boussinesq equations (Fig. 43c). Over significant ε-ranges defects appeared in the square lattice, and for some parameters the "lattice" was really destroyed; but local four-fold coordination persisted for ε ≤ 0.13 over the range 70 ≤ Ω ≤ 250. For larger ε the pattern was more nearly reminiscent of
the KL-state. The occurrence of squares at onset in this system is completely unexpected and not predicted by theory; according to the theory (Clever & Busse, 1979; Chue & Knobloch, 1993) the KL-instability should continue to be found near onset also at these higher values of $\Omega$. Thus we are faced with a major disagreement with theoretical predictions in a parameter range where one might have expected the theory to be reliable. Additional experiments and simulations in the range $0 \lesssim \Omega \lesssim 400$ and $0.7 \lesssim \sigma \lesssim 5$ clearly should be carried out.

![Figure 44: The rotation rate of the square patterns formed with water ($\sigma = 5.4, \Omega = 170$) as a function of $\varepsilon$. After Bajaj et al (1998).](image)

A further interesting aspect of the square patterns is that the lattice rotates slowly relative to the rotating frame of the apparatus. This was found in the experiments with Argon and water as well as in the simulation. Figure 44 gives the angular rotation rate $\omega$ (scaled by $d^2/\nu$) of the lattice for the water experiment. The data are consistent with $\omega(\varepsilon)$ vanishing as $\varepsilon$ goes to zero. Thus the bifurcation to squares is not a Hopf bifurcation. Presumably, as the aspect ratio of the cell diverges, the slope of $\omega(\varepsilon)$ vanishes since an infinitely extended lattice can not rotate. Alternatively, of course, the lattice might become unstable as $\Gamma$ becomes large. It will be interesting to study the $\Gamma$ dependence of $\omega$ experimentally.

## 9 CONCLUSION AND OUTLOOK

Convection in gases has opened up new vistas and brought a number of advantages for the study of pattern formation. First, the time scales of pattern evolution are typically two orders of magnitude faster than, for instance, for water. Second, flow visualization by the shadowgraph method has very high sensitivity for these thin layers (see de Bruyn et al (1996)). Third, the small thickness has brought us large-aspect-ratio cells which made it possible to study phenomena such as SDC which do not occur in smaller systems. Fourth, these thin layers are more susceptible to the influence of thermal noise than is the case for classical liquids, thus enabling the quantitative study of noise-induced fluctuations. Fifth, the conductivity of gases usually is very small and thus it is relatively easy to satisfy the condition (assumed in much of the theoretical work) that the conductivities of the top and bottom plates are much larger than that of the fluid. And finally, the gas mixtures have afforded access to Prandtl numbers as small as 0.16 without inhibiting flow visualization. Because of these unique aspects, there are a number of additional as yet unexploited opportunities provided by gas
convection, and in this section we mention briefly a few particularly interesting problems which are accessible to experiment.

![Graph showing Prandtl number as a function of mole fraction](image)

Figure 45: The Prandtl number $\sigma$ as a function of the mole fraction $X$ of the heavy component for four gas mixtures at a pressure of 22 bar and at 25 $^\circ$C. After Liu & Ahlers (1997).

9.1 RBC with rotation at small $\sigma$

The rotating RBC system becomes particularly interesting for $\sigma < 1$. Prandtl numbers as small as 0.16 can be reached in mixtures of gases if one component has a molecular weight which is much larger than that of the other (Liu & Ahlers, 1997). This is illustrated in Fig. 45, which gives $\sigma(x)$ as a function of the mole fraction $x$ of the heavier component for four mixtures. An important question in this relation is whether the mixtures will behave similarly to pure fluids with the same $\sigma$. To a good approximation this is expected to be the case because the Lewis numbers (the ratios of the mass diffusivities to the thermal diffusivities) are of order one. This means that heat diffusion and mass diffusion occur on similar time scales. In that case, the effect of the concentration gradient will be primarily to contribute to the buoyancy force in synchrony with the thermally-induced density gradient, and thus for $\Psi > 0$ ($\Psi$ is the separation ratio of the mixture) the critical Rayleigh number will be reduced. Scaling bifurcation lines by $R_c(\Psi)$ will mostly account for the mixture effect. To some extent this was shown already by experiment (Liu & Ahlers, 1996, 1997) Recent additional measurements (KMS Bajaj, W Pesch, and G Ahlers unpublished) demonstrated that the bifurcation line $R_c(\Psi, \Omega)/R_c(\Psi, 0)$ and critical wavenumber $q_c(\Psi, \Omega)/q_c(\Psi, 0)$ are within experimental resolution independent of $\Psi$. In addition, linear stability analyses for these mixtures (KMS Bajaj, W Pesch, and G Ahlers unpublished) also showed that these ratios are only very weakly dependent upon $\Psi$.

For $\sigma > 0.33$ the primary bifurcation at easily accessible rotation rates is predicted to be stationary and supercritical. At very large $\Omega$ and for $\sigma < 0.68$ it is predicted to be preceded by a supercritical Hopf bifurcation (Chue & Knobloch (1993). In the stationary case one always expects KL-chaos. As discussed above in Sec. 8.3, experiments do not agree with this; for $\Omega \approx 70$ square patterns, which are clearly unrelated to the typical KL-domains, appear near onset. The range
0.16 ≤ σ ≤ 0.33 is truly remarkable because of the richness of the bifurcation phenomena, which occur there when the system is rotated. For instance, for σ = 0.26 there is a range from Ω ≈ 16 to 190 over which the bifurcation is predicted to be subcritical. This is shown by the dashed section of the curve in Fig. 46B. The subcritical range depends on σ. In Fig. 46A it covers the area below the dashed curve. Thus, the dashed curve is a line of tricritical bifurcations. It has a maximum in the Ω − σ plane. An analysis of the bifurcation phenomena which occur near it in terms of Landau equations may turn out to be interesting. One may expect path-renormalization (Fisher, 1968) of the classical exponents in the vicinity of the maximum. We are not aware of equivalent phenomena in equilibrium phase transitions, although presumably they exist in as yet unexplored parameter ranges. The Hopf bifurcation which precedes the stationary bifurcation at relatively large Ω is predicted to be supercritical and to lead to standing waves of convection rolls (Clune & Knobloch, 1993). Standing waves are relatively rare; usually a Hopf bifurcation in a spatially-extended system leads to traveling waves. The bifurcation lines for σ = 0.26 are shown in Fig. 46B. As can be seen there, the Hopf bifurcation terminates at small Ω at a codimension-two point on the stationary bifurcation which, depending on σ, can be super- or sub-critical. The line of codimension-two points is shown in Fig. 46A as a

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Footnote: Early theoretical evidence for the existence of a subcritical and tricritical bifurcation is contained in the work of Clever & Busse (1979). More recent calculations of the tricritical line by Clune & Knobloch (1993) are inconsistent with the result of Clever & Busse (1979). A new calculation (KMS Bajaj, W Pesch, G Ahlers, unpublished) of the tricritical line yielded the results shown in Fig. 46 which are more detailed than, but agree with, those of Clever & Busse (1979).
dash-dotted line. One sees that the tricritical line and the codimension-two line meet at a codimension-three point, located at \( \Omega \simeq 270 \) and \( \sigma \simeq 0.24 \). We note that this is well within the parameter range accessible to experiments with gas mixtures. We are not aware of any other experimentally-accessible examples of codimension-three points. This particular case should be accessible to analysis by weakly-nonlinear theories, and a theoretical description in terms of Ginzburg-Landau equations would be extremely interesting and could be compared with experimental measurements.

9.2 RBC with inclination

Another interesting variation of traditional RBC is convection in a fluid layer which is inclined relative to gravity. Then, as a function of Prandtl number and inclination angle, not only buoyancy driven but also shear-flow driven instabilities occur. As in RBC with rotation, the system is particularly rich in the Prandtl-number range accessible in experiments with compressed gases.

![Sketch of the geometrical configuration and the shear-flow velocity profile.](image)

Figure 47: Sketch of the geometrical configuration and the shear-flow velocity profile.

A summary of the pertinent literature can be found in the papers by Busse & Clever (1992) and Kelly (1994). In contrast to usual RBC, the basic state consists of heat conduction and a parallel shear flow with cubic velocity profile. A schematic plot is given in Fig. 47. The shear flow breaks the in-plane isotropy of the usual RBC. In this sense the system is similar to convection in liquid crystals with planar alignment of the director (see, for instance, Kramer & Pesch (1995)). For angles smaller than 90° the fluid is heated from below, and both buoyancy and shear are destabilizing, while for angles above 90° the fluid layer is heated from above, where buoyancy is stabilizing and shear is destabilizing.

Linear stability analysis of the basic state is discussed in the papers by Fujimura & Kelly (1993a,b) and Chen & Pearlstein (1989). Depending on the inclination angle \( \gamma \) and on \( \sigma \), longitudinal, oblique, transverse, and travelling transverse rolls are the possible flow structures at onset. The most common pattern for \( \gamma < 90^\circ \) and \( \sigma > 0.6 \) is longitudinal rolls with their axes aligned with the shear-flow (\( q = (\alpha, 0) \)). This situation is analogous to Rayleigh-Bénard convection with Poiseuille flow for which longitudinal rolls is the preferred state (see e.g., Fujimura & Kelly, 1995). The onset of longitudinal rolls is independent of \( \sigma \). As shown by Kurzweg (1970) for the linearized equations and later by Clever (1973) in the nonlinear case, longitudinal rolls can be described by a suitable rescaling of any two-dimensional solution of the horizontal Rayleigh-Bénard problem. The
instability of the ground state with respect to longitudinal rolls is at the critical Rayleigh number $R_c^L = R_c / \cos \gamma$ and the critical wavenumber $q_c^L = \alpha_c$, where $R_c = 1708$ and $\alpha_c = 3.117$ are the critical Rayleigh and wavenumbers of horizontal RBC. The analysis of Geshami & Zhukovskii (1969) showed that oblique rolls with $q = (\alpha, \beta)$ always have a higher threshold compared to longitudinal rolls. The critical Rayleigh numbers of the other solutions are not so easy to obtain, and a detailed discussion is given by Fujimura & Kelly (1993a). For Prandtl numbers $0 < \sigma < 0.26$, transverse rolls with $q = (0, \beta)$ (roll axes transverse to the shear-flow) are realized for all angles of inclination. For $\sigma > 0.26$ and $0^\circ < \gamma < \gamma_c = 77.5^\circ$, longitudinal rolls are the linear perturbation that goes unstable first. For $0.26 < \sigma < 12.42$ and $\gamma > \gamma_c$, transverse rolls have a lower threshold. For $\sigma > 12.42$, travelling transverse rolls are realized at onset for angles close to $90^\circ$. The codimension-two point, at $\gamma_c$, where transverse rolls and longitudinal rolls bifurcate at the same threshold, can be understood in the context of a competition of the two different physical instability mechanisms: the thermal, leading to longitudinal rolls, and the hydrodynamic (shear-flow), leading to transverse rolls. With increasing $\sigma$ the codimension-two point moves quickly to angles close to $90^\circ$. Above $90^\circ$ the fluid layer is heated from above and the instabilities are shear-flow driven. In Fig. 48 the stability diagram of the basic state is plotted for three different Prandtl numbers. Also shown in Fig. 48 is the experimentally measured onset for pressurized CO$_2$ gas with $\sigma = 1.06$. The theoretical predictions agree well with the experimental values. However, above $\gamma_c$ travelling transverse rolls were observed in the experiment while the theory predicts stationary transverse rolls. Perhaps this can be attributed to NOB effects, but only further investigations will settle this issue. Meanwhile,
the results of Daniels et al (1999) are the first experimental confirmation of the theoretically predicted onset values for angles from $0^\circ$ to $120^\circ$.

Figure 49: Snapshot of the convective pattern for $\gamma = 120^\circ$, $\sigma = 1.06$ in a cell of aspect-ratio $21 \times 42$. The higher (lower) end of the inclined cell is marked with “up” (“down”). From KE Daniels and E Bodenschatz, unpublished.

The nonlinear regime and the possible flow structures have been studied theoretically in two- and three-dimensional simulations by Clever & Busse (1977), Buse & Clever (1992), and with semi-analytical methods by Auer (1993) and Fujimura & Kelly (1993a,b). The analysis by Fujimura & Kelly (1993a,b) focuses on the interesting region at the codimension-two point where both longitudinal and transverse rolls bifurcate at onset. They predict a bimodal state which was found recently (KE Daniels and E Bodenschatz, unpublished), Clever & Busse (1977), Buse & Clever (1992), and Auer (1993) investigated the nonlinear regime in the region where longitudinal rolls appear at threshold. One of their predictions is that for a convecting gas with $\sigma \approx 1$ longitudinal rolls should be unstable to a pattern of stationary undulations.

Previous experimental investigations of convection in inclined layers are summarized by Shadid & Goldstein (1990). The experiments were performed in rectangular cells with the largest aspect ratio being $20 \times 40$. In most cases the heat flux across the layer was measured. In the experiments by Hart (1971a,b), Ruth et al (1980), and Ruth (1980) the fluid flow was visualized by introducing foreign particles into it. When the fluid was air, smoke was used; and for water ground fish scales were employed. As a compromise to allow visualization, one of the well-conducting boundaries (metal top plate) was replaced with a transparent but poorly-conducting plate (glass or PMMA). The results should only be considered as qualitative, since it is well known from convection in a horizontal layer that in the case of asymmetric boundaries the bifurcation diagram near onset is considerably modified (see e.g. Buse, 1989). In the recent experiment by Shadid & Goldstein (1990) symmetric boundaries with low heat conduction were used.
A liquid-crystal sheet on the cell bottom and a glass top-plate were used for visualization of the temperature field. As explained and observed by the authors, insulating top and bottom boundaries lead to rectangular patterns at the onset of convection. This situation is again different from that addressed in the theoretical calculations. Experiments and numerical simulations by Kunhardt & Oertel (1988) address the problem of convection in an inclined rectangular box of aspect ratio 1:4:10 and 1:2:4. They visualized the flow by optical interference techniques from the side. They showed that the flow in these very confined geometries is strongly influenced by the sidewalls. Again, close to onset the theoretical results discussed before are not applicable to this situation.

In summary, all experiments so far have been limited by the small aspect ratios of the cells, the unequal or weakly conducting cell boundaries, the methods of flow visualization, or the e-resolution of the experiments. In the linear regime good qualitative agreement for the onset of longitudinal and transverse rolls was found. In the nonlinear regime the transitions to wavy rolls and to time dependent patterns were confirmed qualitatively. However, the experiments gave results that can only be compared semi-quantitatively to the recent theoretical predictions by Clever & Busse (1977), Busse & Clever (1992), Auer (1993), and Fujimura & Kelly (1993a,b).

By using compressed gases as a convecting fluid, it is now possible to explore inclined RBC for a wide range of Prandtl numbers in large-aspect-ratio systems. For example, Daniels et al (1999) investigated the phase diagram for $\sigma \approx 1$ and found a variety of novel instabilities which are still under investigation. One important discovery is that the instability of longitudinal rolls is not to stationary undulations as predicted by Clever & Busse (1977), but to spatio-temporal chaos. New numerical simulations on the basis of Sec. 3.4 by Daniels et al (1999) also found this spatio-temporal chaotic state. Daniels et al (1999) were also able to visualize the convective patterns when the experiment was heated from above. An example is shown for $\gamma = 120^\circ$ in Fig. 49 where the flow is purely shear driven.

9.2 RBC with modulation of the vertical acceleration

The response of hydrodynamic systems to periodic modulation of the driving has been of considerable interest since the seminal experiments of Donnelly et al (1962) The case of modulated RBC has received much theoretical attention (Ahlers et al, 1985b). Either the vertical acceleration or the temperature difference can be modulated. The consequences are somewhat different for two reasons. First, the acceleration affects the momentum-balance equation, whereas the temperature enters into the heat equation. Second, gravity modulation (a body force) does not change the symmetry of the conduction state whereas temperature modulation (when applied to the top or bottom plate) induces a nonlinear conduction profile and thus leads to a transcritical bifurcation and hexagonal patterns (Meyer et al, 1988, 1992). In either case, the simplest modulation would take the form $\epsilon(t) = \epsilon_0 + \delta \sin(\omega t)$, with $\omega$ and $t$ scaled by the vertical thermal diffusion time $\tau = \delta^2/\kappa$. Among the expected effects are a shift in the threshold, a subharmonic bifurcation over certain ranges of $\delta$ and $\omega$, and changes in the nonlinear properties such as the Nusselt number and the patterns above onset. Experiments using thermal modulation (Meyer et al, 1988, 1992) have been diffi-

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26A review of early work is given by Davis (1957).
cult and confined to rather restricted regions of the $\omega - \delta$ parameter space. The interesting region of large $\delta$ was difficult to reach because of the large thermal mass of the bottom plate of the convection cell. Gravity modulation seemed even less promising with conventional convection systems employing water as the fluid because $\tau_v$ is of the order of $10^2$ seconds and, with $\delta$ of any significant size and $\omega$ small enough to produce interesting effects, implies unrealistic periodic vertical displacements. Convection in gases has much to offer in this area. The fluid layers are much thinner, with thermal diffusion times two orders of magnitude smaller than in water (i.e. near 1 sec). Consequently the desired vertical displacements are realistically achievable in these systems. In addition, the Prandtl number is near one, where modulation effects are maximal. We are pleased to see that work on this interesting problem is now under way (Rogers et al., 1998).

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