Reynolds-number measurements for low-Prandtl-number turbulent convection of large aspect-ratio samples

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We present experimental results for the Reynolds number $\text{Re}_U$ based on the horizontal mean-flow velocity $U$ and for $\text{Re}_V$ based on the root-mean-square horizontal fluctuation velocity $V$ for turbulent Rayleigh-Bénard convection in a cylindrical sample of aspect ratio $\Gamma = 10.9$ over the Prandtl number range $0.18 \leq \text{Pr} \leq 0.88$. The results were derived from space-time cross-correlation functions of shadowgraph images, using the elliptic approximation of He & Zhang (2006). The data cover the Rayleigh number range from $3 \times 10^5$ to $2 \times 10^7$. We find that $\text{Re}_U$ is nearly two orders of magnitude smaller than the values given by the Grossmann-Lohse (GL) model (Grossmann & Lohse 2002) for $\Gamma = 1.00$ and attribute this difference to averaging caused by lateral random diffusion of the LSC cells in large-$\Gamma$ samples. For the fluctuations we found $\text{Re}_V = \tilde{R}_0 \text{Pr}^\alpha \text{Ra}^\eta$, with $\tilde{R}_0 = 0.31$, $\alpha = -0.53 \pm 0.11$, and $\eta = 0.45 \pm 0.03$. That result agrees well with the GL model. The close agreement of the coefficient $\tilde{R}_0$ must be regarded as a coincidence because the GL model was for $\Gamma = 1.00$ and for a mean-flow velocity $U$.

1. Introduction

Convection in a fluid confined between parallel horizontal plates and heated from below (Rayleigh-Bénard convection or RBC) occurs over a wide range of the Rayleigh number $\text{Ra} = \alpha g \Delta T L^3 / \kappa \nu$ and the Prandtl number $\text{Pr} = \nu / \kappa$. Here $\alpha$ is the isobaric thermal expansion, $g$ is the gravitational acceleration, $\Delta T$ is the temperature difference between the plates, $L$ is the spacing between the plates, $\kappa$ is the thermal diffusivity, and $\nu$ is the kinematic viscosity. Physically realizable studies use cells with finite lateral extent. Thus, the aspect ratio $\Gamma \equiv D / L$ of the sample is an additional relevant parameter. Here $D$ is the lateral size of the sample, for instance the diameter of a cylindrical cell.

Much of the previous work on RBC can be broadly divided into two areas: pattern formation (Cross & Hohenberg 1993; Bodenschatz et al. 2000), characterized by low $\text{Ra}$ ($\lesssim 10^5$) and large $\Gamma$ ($\gtrsim 5$), and turbulent convection (Ahlers 2009; Ahlers et al. 2009; Lohse & Xia 2010), characterized by high $\text{Ra}$ [up to $10^{17}$ (Niemela et al. 2000; Chavanne et al. 2001; He et al. 2012)] and generally small $\Gamma$ ($\mathcal{O}(1)$ or less). Increasing $L$ helps to reach higher Rayleigh numbers, as $\text{Ra} \sim L^3$, but practical constraints limit the lateral extent of the cell, hence the prevalence of smaller aspect ratios in high Rayleigh number work. In the present investigation we explore the relatively large $\Gamma \approx 11$, but still reach relatively large values of $\text{Ra}$ (up to $2 \times 10^7$) where the flow is turbulent. We accomplish this by using compressed gases at elevated pressures.

The turbulent fluid flow in small-$\Gamma$ cells for $\text{Ra} \gtrsim 10^6$ is highly fluctuating, but in the time average tends to be organized as a single convection roll or possibly two rolls with
one located above the other [see Xi & Xia (2008); Weiss & Ahlers (2011); and references therein]. The strength and azimuthal diffusion of this large-scale circulation display a rich dynamics (Brown et al. 2005; Brown & Ahlers 2008). A dimensionless parameter related to the fluid flow is the Reynolds number

\[ \text{Re}_U = \frac{UL}{\nu}. \]  

(1.1)

Here \( U \) is a time averaged characteristic velocity, for instance the maximum velocity found in the LSC which occurs near (but not too near) the side walls or the plates (Qiu & Tong 2001). Good agreement has been reported between some (but not all) experimental measurements of variously defined Reynolds numbers for \( Pr \approx 4 \) and larger for some \( Ra \) ranges and \( \Gamma \) near one [see, for instance, Qiu & Tong (2001); Lam et al. (2002); Brown et al. (2007); Xia (2007)] and a theoretical model of Grossmann & Lohse (2002) (the GL model; we shall refer to the Reynolds numbers given by that model as \( \text{Re}_{GL} \)). We note that the GL model contains a parameter which was adjusted so that the overall magnitude of \( \text{Re}_{GL} \) agreed with experimental results for \( \text{Re}_U \) measured in a sample with \( \Gamma = 1.00 \) and \( Pr = 5.4 \) (Qiu & Tong 2001). Although the model is expected to give the right dependence of \( \text{Re} \) on \( Pr \) and \( Ra \) which is expected to be the same for all \( \Gamma \), it should not be expected that the magnitude of \( \text{Re}_{GL} \) will agree with data for very different \( \Gamma \). Nor can it be expected to reproduce the magnitude of Reynolds numbers based on differently defined velocities, such as the root-mean-square (rms) fluctuation velocity \( V \) for instance or the vertical average over the sample that is measured by the shadowgraph method (see Sec. 2.3 below).

Also of interest is the flow behavior for systems with larger \( \Gamma \). There the influence of the walls is less and the experimental systems are better analogs to various astrophysical and geophysical phenomena [see, for instance, Howard & LaBonte (1980); Rahmstorf (2000)]. For large-\( \Gamma \) systems the LSC takes the form of many cells distributed over the horizontal plane (Hartlep et al. 2005; Bailon-Cuba et al. 2010; Bosbach et al. 2012). Under the influence of vigorous turbulent fluctuations these cells experience a random lateral dynamics. Thus we would expect that, in the time average, the value of \( U \) measured at any fixed location in the sample interior will vanish or become quite small. Here it is interesting to note that the time averages obtained by DNS still revealed a large-scale flow structure (Bailon-Cuba et al. 2010). This is so because the time average used in the DNS was sufficiently short to permit the structures to survive. We find that the images, averaged over the duration of our experiments, reveal virtually no structure for our sample with \( \Gamma \approx 11 \), and that the values for \( \text{Re}_U \) of the averages are about two orders of magnitude smaller than \( \text{Re}_{GL} \). However, a larger rms fluctuation velocity \( V \) characteristic of the turbulent state remains, with an associated Reynolds number

\[ \text{Re}_V = \frac{VL}{\nu}. \]  

(1.2)

In the present paper we focus primarily on \( \text{Re}_V \), where \( V \) is a horizontal component of the rms velocity fluctuations. Our results for this quantity did not differ very much from \( \text{Re}_{GL} \). We were surprised by this result because, as mentioned above, the amplitude \( R_0 \) of the power law \( \text{Re} = R_0 \text{Ra}^{\eta} \) is expected to depend on \( \Gamma \) and on the particular definition of \( \text{Re} \) that is investigated. Thus, the approximate agreement of the amplitude of \( \text{Re}_{GL} \) with our result for the amplitude of \( \text{Re}_V \) for a sample with \( \Gamma >> 1 \) must be regarded as a coincidence. As implied above, the agreement of \( \eta \) with the GL model is of course expected.

Another feature of our work is that we study the nature of the flow for smaller Prandtl
numbers than are usually encountered. We used gas mixtures (Liu & Ahlers 1997) to access Pr as small as 0.18, much lower than accessible with commonly used fluids such as pure gases at room temperature ($0.67 \leq Pr \lesssim 1$) and water or other liquids ($Pr \gtrsim 2$). Liquids metals have $Pr \lesssim 0.07$ (Lappa 2010), but are not well suited for flow visualization. No pure fluids have $Pr$ between about 0.07 and 0.67, so gas mixtures are the only means we know of to access part of this Pr range.

Our measurements of $Re_U(Ra, Pr)$ and $Re_V(Ra, Pr)$ for $0.18 < Pr < 0.9$ and $\Gamma = 10.9$ were made by applying the Elliptic Approximation (EA) of He and Zhang (He & Zhang 2006; Zhao & He 2009) to space-time cross-correlation functions of shadowgraph images. This technique had not previously been applied to images, as far as we know. Section 2 describes the experimental apparatus, the working fluids, the shadowgraph images, and the EA applied to the image correlation functions to obtain the mean velocity $U$ (which, as expected, turns out to be small) and the horizontal component of the rms fluctuation velocity $V$. Section 3 presents the Reynolds numbers for the velocities obtained from the image correlations, and the Rayleigh- and Prandtl-number dependences of these Reynolds numbers. Finally, Section 4 summarizes our results. An Appendix discusses the Ra range over which flows may be considered turbulent as a function of Pr.

2. Experiment and data analysis

2.1. Apparatus

The apparatus was used before and has been described in detail (Xu et al. 2000; Ahlers & Xu 2001; Nikolaenko et al. 2005; Funfschilling et al. 2005; Zhong et al. 2009). For the present work a cylindrical cell had a height $L = 0.93$ cm and a diameter $D = 10.16$ cm. The bottom plate was composed of a sapphire disk 0.318 cm thick which was epoxied on top of a 3 cm-thick copper plate. The top of the sapphire had a surface coating of gold to produce a mirror finish. The power to heat the bottom plate came from a film heater glued to the bottom of the copper plate. The top plate was a sapphire disk with thickness 1.91 cm, which was cooled by water jets directed onto its top surface and which permitted optical access to the sample. The walls of the cell were formed by a steel sidewall, and a spacer ring which set the distance between the top and bottom plates. Gas entered the cell through a small hole in the bottom plate, which connected the cell to a gas-supply manifold via a steel capillary. The cell was in an air and foam filled can inside a temperature-controlled recirculating water bath. The water in the bath was regulated at the desired top plate temperature using a calibrated thermistor in the bath near the top plate and a heater in a feedback loop. We used a shadowgraph apparatus (de Bruyn et al. 1996) for flow visualization.

2.2. Working fluids

We used nitrogen, sulfur hexafluoride, hydrogen–xenon mixtures, or helium–sulfur hexafluoride mixtures as the fluid. The fluid pressures were in the range from 10 to 32 bars. The pressure was held constant to better than 0.1% over the course of a given run.

The various properties of the gases used were calculated with programs based on experimental and theoretical results from the literature (Uribe et al. 1990; Boushehri et al. 1987; Hilsenrath et al. 1960; Gracki et al. 1969; Kestin & Imaishi 1985; Hoogland et al. 1985; Braker & Mossman 1974; Oda et al. 1983; Liu & Ahlers 1997). The results from the pure-gas programs were in good agreement with published values from NIST’s REFPROP program (Lemmon et al. 2007). The properties of the gas mixtures used here were calculated with programs from Liu & Ahlers (1997). For H$_2$–Xe mixtures the quoted uncertainties, based on comparison with available published experimental
results, were 1% for the density, 2% for the viscosity and thermal conductivity. However, the measured thermal conductivity found by Liu & Ahlers (1997) was up to 5% larger than the calculated value, so the results presented therein used the measured thermal conductivity for H$_2$–Xe. For the present results, we multiplied the calculated thermal conductivity by 1.047 to bring it in line with the measured results. For He–SF$_6$ mixtures the quoted uncertainties were estimated to be 2-7% for the density and 10% for the thermal conductivity and heat capacity. The measured thermal conductivity found by Liu & Ahlers (1997) agreed within 3% or so with the calculated value, so here we used the calculated values for He–SF$_6$ unchanged.

2.3. Image acquisition and analysis

The shadowgraph tower (de Bruyn et al. 1996) contained a monochrome Retiga 1300 (QImaging) CCD camera. The camera was capable of 1280 × 1024 pixels and 12-bit pixel intensities, though to increase the frame rate we used smaller images with 8-bit pixel intensities. We used square images approximately centered in the cell. The image size was either 240 × 240 or 16 × 16 pixels, with frame rates between about 13 and 39 images/second and a spatial resolution of 1/d = 2.5 to 4.3 pixels/mm (d/L = 0.43 to 0.25). The results did not depend significantly on the image resolution or frame rate. Either 1024 or 10240 images were obtained for each run. Background division (de Bruyn et al. 1996) was applied to the images to account for non-uniform illumination of the cell. The background image was the average of all the images for a given run. Two typical examples of divided and re-scaled images are shown in Fig. 1. These images were taken with higher spatial resolution than used for the analysis in order to improve their visual appearance.

The primary results, presented below in Secs. 3.1 and 3.2, are for the background-divided images with no other image processing applied. They contained both light and dark structures, which are expected to correspond to regions of cold and hot fluid respectively. However, in Sect. 3.3 we also considered the cases where the images were processed to only include either the light or the dark structures, by selecting a threshold value based on the histogram $H(I)$ of pixel intensities $I(x,y)$. Eight-bit images were used, so the intensity values varied between 0 and the maximum intensity $I_{max}$ in 255 increments. To choose the threshold values, the location $I_{peak}$ of the maximum of $H(I)$ was determined first. Then the threshold intensity $I_{thresh}$ was chosen such that the area under the histogram between $I_{peak}$ and $I_{thresh}$ equaled some specified fraction $F$ of the total area between $I_{peak}$ and either $I_{max}$ or $I_{min}(=0)$ as appropriate:

$$F = \frac{\int_{I_{thresh}}^{I_{peak}} H(I)dI}{\int_{I_{peak}}^{I_{max, min}} H(I)dI}.$$  

(2.1)

We used $F = 0.85$, though similar values gave essentially equivalent results. The thresholded image had all points above (below) the threshold value set equal to $I_{thresh}$, and the points below (above) the threshold unchanged.

The shadowgraph intensity $I(x,y)$ is proportional to a vertical average of the horizontal Laplacian of the refractive index field (Rasenat et al. 1989; de Bruyn et al. 1996; Trainoff & Cannell 2002); the refractive index in turn is a linear function of the temperature. The temperature is a passive scalar and follows the local flow everywhere in the cell except near the thermal boundary layers (see, for instance, He et al. (2011) and references therein). Thus temperature measurements yield the same correlation functions as velocity measurements. From very extensive work in the field of pattern formation and chaos (see, for instance, de Bruyn et al. (1996); Bodenschatz et al. (2000)) it is well established that
the shadowgraph images are an excellent representation of the corresponding temperature and/or velocity patterns. It is thus reasonable to assume that the shadowgraph images will also behave as a passive scalar and yield the same correlation function as the velocity.

One difference between the shadowgraph method and local temperature measurements remains. The shadowgraph method averages vertically over the sample height while previous temperature measurements (He et al. 2010; He & Tong 2011; He et al. 2012) have been local. However, lateral displacements of structures survive the vertical averaging of the shadowgraph method. Although contributions from all vertical levels enter into the average (see, for instance Fig. 1), large contributions come from hot and cold plumes while these reside close to the bottom and top plates respectively and move laterally with the prevailing flows. To our knowledge there have been no precise measurements of the distribution function of the vertical location of plumes during that stage of their existence, but they are likely to reside in the bulk rather than in the BLs where they were born. We believe that this is so because they are free to move with any large-scale flow, as can be seen in the movies submitted with this paper as supplementary material, and as has been amply demonstrated before (see, for instance, Funfschilling et al. (2008)). If the plumes resided in the viscous boundary layer, then one would expect their lateral motion to be inhibited. In conclusion, we expect the plumes and other excitations to be residing primarily in the bulk of the sample where they may follow the flow and act as passive scalars. Thus the shadowgraph correlation function $C_I(r, \tau)$, temperature correlation function $C_T(r, \tau)$, and velocity correlation function $C_v(r, \tau)$ should have the same behavior (He et al. 2010). We will use them interchangeably below. Experimental support for the validity of this approach comes from the agreement between the Reynolds numbers deduced from three different methods of image processing which emphasize different vertical portions of the sample (see Figs. 1 and 7).

The shadowgraph-intensity space-time cross-correlation function is given by

$$C_I(r, \tau) = \frac{\langle \delta I(x + r, t + \tau) \delta I(x, t) \rangle_{t,x}}{\langle \sigma_I \rangle_1 \langle \sigma_I \rangle_2}$$

(2.2)

where $\langle \ldots \rangle_{t,x}$ denotes a space and time average, $\delta I = I - \langle I \rangle_t$, and $(\sigma_I)_1$ and $(\sigma_I)_2$ are the
standard deviations of $\delta I$ at $x$ and $x + r$ respectively. Assuming translational invariance of the sample and sufficiently long time series to yield adequate statistics, one expects $(\sigma_I)_1$ and $(\sigma_I)_2$ to be equal to each other, but in practice each was evaluated separately at its location.

The correlation function $C(r, \tau)$ for a given run was calculated by averaging the individual $C_j(r, \tau), j = 1, \ldots, N_I$, for each of the $N_I = 1024$ or 10240 images from that run. The time shift $\tau$ was done by comparing the $i$-th image with the $(i + \tau f)$-th image in the sequence, where $f$ is the framerate in images per second. The spatial shift $r$ was done by comparing a given pixel at $(x_1, y_1)$ in each image with pixels at $(x_1 + r, y_1)$ and $(x_1, y_1 + r)$ in a second image, and repeating the procedure for all pixels in each image. The results for all pixels in all images in a particular run were averaged to obtain a single value of $C(r, \tau)$ for each $r$ and $\tau$. Thus our analysis was in a sense one-dimensional, averaging the components $V_x$ and $V_y$ of $V$ in the $x$ and $y$ directions. An analysis using only one direction showed that $V_x$ and $V_y$ were equal to each other within statistical errors; using their average yielded more precise results. Both $r$ and $\tau$ were restricted to some range, generally $\pm 5$ or 15 images or pixels. Typically this corresponded to a physical range of up to $\pm 6$ mm for $r$ and up to $\pm 1.2$ seconds for $\tau$. The shape of $C(r, \tau)$ closest to the peak at $r = 0$, $\tau = 0$ was most important, and $C(r, \tau)$ decreased quickly from the peak, so a wider range of $r$ and/or $\tau$ was not necessary.

2.4. The elliptic approximation (EA)

We measured the Reynolds number using space and time cross-correlations of successive shadowgraph images. The turbulent flow and fluctuations in the cell caused the visible structures to undergo both spatial translation and deformation. We used the recently developed Elliptic Approximation (He & Zhang 2006; Zhao & He 2009) (EA) to analyze this dynamics. The EA consists of a systematic Taylor-series expansion of the space-time correlation function to second order. This expansion can be used to determine both the mean velocity $U$ and the rms fluctuation velocity $V$. This method was previously applied to RBC with Pr near 5 using local temperature measurements for samples with $\Gamma = 1$ and Ra near $10^{10}$ (He et al. 2010; He & Tong 2011) and with Pr $\approx 0.8$ and $\Gamma = 1/2$ for Ra up to $10^{15}$ (He et al. 2012). It was also used by Zhou et al. (2011) to analyze particle-image velocimetry data for a $\Gamma = 1.00$ sample and Pr $\approx 5$ for Ra over the range from $6 \times 10^9$ to $10^{11}$. Here we apply the EA to a large array of image pixels, instead of to a small number of discrete temperature sensors, over a range of Pr from 0.18 to 0.9 and Ra from $3 \times 10^5$ to $2 \times 10^7$. Details of the computation of $C(r, \tau)$ were given in the previous section.

The Taylor-series expansion of $C(r, \tau)$ has the form

$$C(r, \tau) = C(0, 0) + \frac{\partial C(0, 0)}{\partial r}r + \frac{\partial C(0, 0)}{\partial \tau}\tau + \frac{\partial^2 C(0, 0)}{\partial r \partial \tau}r\tau + \frac{1}{2} \left[ \frac{\partial^2 C(0, 0)}{\partial r^2}r^2 + \frac{\partial^2 C(0, 0)}{\partial \tau^2}\tau^2 \right] + \ldots . \quad (2.3)$$

For a spatially homogeneous and statistically stationary sample the two first derivatives vanish (He & Zhang 2006). The EA consists of truncating Eq. 2.3 at second order. This second-order expansion can be written as (He & Zhang 2006; Zhao & He 2009)

$$C(r, \tau) = C(0, 0) + \frac{1}{2} \frac{\partial^2 C(0, 0)}{\partial r^2}r_E^2 \quad (2.4)$$

where

$$r_E^2 = (r - U\tau)^2 + V^2\tau^2 \quad (2.5)$$
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Increasing \( r_E \)
Decreasing \( C(r, \tau) \)

\[ C(r_0, \tau_0) \]

(a)

(b)
(c)

Figure 2. a): Sketch of iso-correlation contours. For a given time shift \( \tau_0 \), the maximum of the correlation function \( C(r, \tau_0) \) occurs at a spatial shift \( r_p(\tau_0) \). The same procedure (not shown) for a given spatial shift \( r_0 \) gives the time shift \( \tau_p(r_0) \) where the maximum of \( C(r_0, \tau) \) occurs. b): Sketch illustrating the measurement of \( U \) and \( V \) from \( C(r_0, \tau) \) using the two-point method in the case where \( U \) is not negligible compared to \( V \). c): Sketch illustrating the measurement of \( V \) from \( C(r_0, \tau) \) using the two-point method in the case where \( U = 0 \) or negligibly small.

with

\[
U = - \frac{\partial^2 C(0, 0)}{\partial r \partial \tau} \left[ \frac{\partial^2 C(0, 0)}{\partial r^2} \right]^{-1}
\]

and

\[
V^2 = \frac{\partial^2 C(0, 0)}{\partial \tau^2} \left[ \frac{\partial^2 C(0, 0)}{\partial r^2} \right]^{-1} - U^2.
\]

The contribution \( U \) is the time-averaged mean-flow velocity which for our case is expected to be small or zero, and \( V \) is dominated by a contribution from a rms fluctuation velocity \( v_0 \) with \( v_0^2 = 2 \int E(k)dk \) where \( E(k) \) is the energy spectrum of the velocity (Zhao & He 2009). A small additional contribution to \( V \) is proportional to the local shear times the Taylor length microscale (Zhao & He 2009), which in our case is also expected to be negligible or absent. Thus we expect our measurements of \( V \) to be dominated by the velocity fluctuations.

2.5. Data analysis based on the EA

One way to determine \( U \) and \( V \) is to use the derivatives of \( r_E(r, \tau) \). One sees from Eq. 2.4 that a constant value of the length \( r_E \) corresponds to a constant \( C(r, \tau) \). Equation 2.5 shows that a constant \( r_E \) [and thus a constant \( C(r, \tau) \)] corresponds to an ellipse in the \( r - \tau \) plane. This is illustrated in Fig. 2a. The values \( r_p \) and \( \tau_p \) can thus be obtained from the extrema of \( r_E \), i.e. by setting the partial derivatives of \( r_E(r, \tau) \) (Eq. 2.5) to zero (He et al. 2010; Zhou et al. 2011):

\[
\frac{\partial r_E}{\partial r} = 0 \quad \rightarrow \quad r_p = U \tau
\]

and

\[
\frac{\partial r_E}{\partial \tau} = 0 \quad \rightarrow \quad \tau_p = \left[ \frac{U}{U^2 + V^2} \right] r .
\]

Values for \( r_p(\tau) \) are obtained by fitting the Gaussian function

\[
C(r, \tau_0) = C_0 \exp[(r - r_p)^2/(2\sigma^2)] + C_1
\]

to values of \( C(r, \tau_0) \) separately for each of all available constant \( \tau_0 \). Similarly, the function

\[
C(r_0, \tau) = C_2 \exp[(\tau - \tau_p)^2/(2\sigma^2)] + C_3
\]
Figure 3. The time auto-correlation function $C(0, \tau)$ (solid circles) and the space-time cross-correlation function $C(r_0, \tau)$ for a spatial displacement $r_0 = 0.35\text{mm}$ (solid squares, red online). The solid and dashed lines are fits of Gaussian functions, Eq. 2.11, to the three points nearest the maxima. For $C(0, \tau)$ $\tau_p = 0$ was fixed, while for $C(r_0, \tau)$ $\tau_p$ was adjusted in the fit. The data are for $\text{H}_2$–$\text{Xe}$, $Ra = 3 \times 10^5$, $Pr = 0.18$.

is fit to values of $C(r_0, \tau)$ separately at each available constant $r_0$ to obtain $\tau_p(r)$. Straight-line fits of Eqs. 2.8 or 2.9 to these data for $\tau_p(\tau)$ and $\tau_p(r)$ then yield the desired results for $U$ and $U/(U^2 + V^2)$. This method works well when both $U$ and $V$ are of comparable size. When, as in our case, $U << V$, we can still determine $U$ from Eq. 2.8, albeit not with very high precision because it is small. However, the determination of $V$ from Eq. 2.9 leads to errors that are larger than desired. Thus, for the determination of $V$ we turn to the slightly different method described next.

Another way to determine $U$ and $V$ utilizes measurements at only two points in space (He et al. 2010; He & Tong 2011). This method usually was employed when measurements were available only at two points, but it has merit also in our case. It is illustrated in Fig. 2b for the case where both $U$ and $V$ are of significant size, and in Fig. 2c for the case where $U = 0$. Its implementation is illustrated in Fig. 3. As illustrated in Fig. 2b, one has

$$C(r_0, 0) = C(0, \tau_0)$$

(2.12)

where the two points $(r_0, 0)$ and $(0, \tau_0)$ lie on the same elliptic contour. As seen in Fig. 3, $\tau_0$ is given by the time displacement between the time auto-correlation function $C(0, \tau)$ when it has the same value as the cross-correlation function $C(r_0, \tau = 0)$. This is illustrated by the double arrow in the figure. Having found $\tau_0$, one can form the effective velocity

$$V_{eff} = r_0/\tau_0$$

(2.13)

This procedure can be followed for any displacement $r_0$ for which data may be available. In our case, one can use any small integer $n$ (in practice only $n = 1$ or 2) times the distance $d$ between adjacent pixels, and the value of $V_{eff}$ is expected to be independent of $n$.

In practice, we fit Eq. 2.11 to $C(r_0, \tau)$ versus $\tau$ at each measured $r_0$. An example was shown above in Fig. 3. The fits were done only to those points close to the peak in order to avoid undue influence from experimental noise in the tails of the distributions. The time increment $\tau_0$ between the fits to $C(0, \tau)$ and $C(r_0, \tau)$ were then determined. We used $r_0 = d$ where $d$ is the spacing between pixels, but the results for $V$ were not significantly
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Figure 4. The Reynolds number of the large-scale circulation \( \text{Re}_U \) as a function of \( \text{Ra} \). Circles (purple online): \( \text{Pr} = 0.18 \). Squares (orange online): \( \text{Pr} = 0.54 \). Diamonds (red online): \( \text{Pr} = 0.74 \). Up-pointing triangles (green online): \( \text{Pr} = 0.81 \). Down-pointing triangles (blue online): \( \text{Pr} = 0.88 \). Dotted line (purple online): \( \text{Re}_{\text{GL}} \) for \( \text{Pr} = 0.18 \). Dashed line (blue online): \( \text{Re}_{\text{GL}} \) for \( \text{Pr} = 0.88 \). Solid line: A power law with the exponent fixed at the value 0.45 of the GL model and the pre-factor fit to the data.

Figure 4 shows \( \text{Re}_U \) determined using Eq. 2.8 as a function of Ra on logarithmic scales. The data show considerable scatter because \( \text{Re}_U \) is remarkably small and for that reason difficult to measure with our techniques. Nonetheless they reveal a trend with Ra. For comparison, the dotted and dashed lines give the GL prediction (Grossmann & Lohse 2002) for \( \text{Re}_U \) for \( \text{Pr} = 0.18 \) and 0.88 respectively, based on the large-scale circulation in a \( \Gamma = 1.00 \) cylindrical sample. As expected, in our spatially extended sample \( \text{Re}_U \) is much smaller. The solid line is a fit of a power law to the data, keeping the exponent at the value 0.45 appropriate for the GL model and adjusting the pre-factor. The result is that on average \( \text{Re}_U \) is typically two orders of magnitude smaller than it is for a \( \Gamma = 1.00 \) sample (Grossmann & Lohse 2002). In principle we might expect that \( \text{Re}_U \) would average to zero for a sufficiently large sample and over a long enough time interval because of the random nature of the LSC dynamics that is expected in such a sample. It is not clear whether the small surviving value of \( \text{Re}_U \) in our sample is due to its finite (albeit large) lateral size or to insufficiently long time series.

3. Results

3.1. Results for \( \text{Re}_U \)

As discussed above and shown below in Fig. 4, for our application we found that \( U \) is typically about two orders of magnitude smaller than \( V \). Thus the part \( (U/V)^2 \) in Eq. 2.14 is much smaller than our experimental errors, and it is well justified to equate \( V_{\text{eff}} \) with \( V \) in our case.

\[ V_{\text{eff}} = V \sqrt{1 + (U/V)^2} . \] (2.14)
Figure 5 shows $R_{eV}$ calculated using Eqs. 2.12 and 2.14 (see Fig. 3) for several values of Pr. A significant Pr dependence is evident. For comparison we give the dotted, solid, and dashed lines, which are the results of the GL model (Grossmann & Lohse 2002) for $Pr = 0.18, 0.54$, and $0.88$ respectively. Quite unexpectedly, the model (which is for $\Gamma = 1.00$ and $Re_U$) comes fairly close to the data for $\Gamma = 10.9$ and the fluctuation-velocity Reynolds number $Re_V$. One would have expected the same slope (corresponding to the same effective exponent), but the closeness also of the magnitude must be regarded as fortuitous.

We fit the power law of the form $Re_V = R_0 Ra^\eta$, where $R_0$ and $\eta$ are constants, to the $Re_V$ results separately for each Pr. The fits were restricted to points where $Ra \geq 3 \times 10^5$, which is the approximate criterion for the flow to be turbulent for any of the Pr values (see the Appendix). The values of the exponent $\eta$ are presented in Table 1. They are in good agreement with the predicted exponent of the GL model $\eta_{GL} = 0.45$ to 0.45 in this $Ra$ range.

Table 1. Exponents $\eta$ from the fit of $Re_V = R_0 Ra^\eta$ separately to the data sets at a given Pr. The last row is from a multi-variable simultaneous fit to data for all Pr, using the equation $Re_V = \tilde{R}_0 Pr^{\alpha}Ra^\eta$. The results for the exponents may be compared with that of the GL model $\eta_{GL} = 0.45$ for this $Ra$ range.

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>Fluid</th>
<th>$P$ (bars)</th>
<th>$L/\nu$ (sec/cm)</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18</td>
<td>H$_2$–Xe (50:50)</td>
<td>25.0</td>
<td>245.4</td>
<td>0.46 ± 0.04</td>
</tr>
<tr>
<td>0.54</td>
<td>He–SF$_6$ (27:73)</td>
<td>14.8</td>
<td>353.6</td>
<td>0.49 ± 0.02</td>
</tr>
<tr>
<td>0.74</td>
<td>N$_2$</td>
<td>32.1</td>
<td>167.9</td>
<td>0.47 ± 0.01</td>
</tr>
<tr>
<td>0.81</td>
<td>SF$_6$</td>
<td>10.5</td>
<td>376.5</td>
<td>0.42 ± 0.04</td>
</tr>
<tr>
<td>-</td>
<td>All of above</td>
<td>-</td>
<td>-</td>
<td>0.45 ± 0.03</td>
</tr>
</tbody>
</table>

3.2. Results for $Re_V$

Figure 5 shows $Re_V$ calculated using Eqs. 2.12 and 2.14 (see Fig. 3) for several values of Pr. A significant Pr dependence is evident. For comparison we give the dotted, solid, and dashed lines, which are the results of the GL model (Grossmann & Lohse 2002) for $Pr = 0.18, 0.54$, and $0.88$ respectively. Quite unexpectedly, the model (which is for $\Gamma = 1.00$ and $Re_U$) comes fairly close to the data for $\Gamma = 10.9$ and the fluctuation-velocity Reynolds number $Re_V$. One would have expected the same slope (corresponding to the same effective exponent), but the closeness also of the magnitude must be regarded as fortuitous.

We fit the power law of the form $Re_V = R_0 Ra^\eta$, where $R_0$ and $\eta$ are constants, to the $Re_V$ results separately for each Pr. The fits were restricted to points where $Ra \geq 3 \times 10^5$, which is the approximate criterion for the flow to be turbulent for any of the Pr values (see the Appendix). The values of the exponent $\eta$ are presented in Table 1. They are in good agreement with the predicted exponent of the GL model $\eta_{GL} = 0.44$ to 0.45 in this $Ra$ and Pr range.

Also shown in table 1 is the value of $\eta$ obtained from a multivariable regression fit of $Re_V(Ra, Pr) = \tilde{R}_0 Pr^{\alpha}Ra^\eta$ simultaneously to all the data, including all Pr values. The fit gave $Re_V = (0.31 \pm 0.14)Ra^{0.45\pm0.03}Pr^{-0.53\pm0.11}$. The effective exponents of the GL model vary somewhat over the parameter ranges of our experiment. While $\eta$ remains in
Figure 6. $\text{Re}_V/\text{Pr}^\alpha$ for $\text{Pr} = 0.18$ (purple circles), $\text{Pr} = 0.54$ (orange squares), $\text{Pr} = 0.74$ (red diamonds), $\text{Pr} = 0.81$ (green upward triangles), and $\text{Pr} = 0.88$ (blue downward triangles). The value for $\alpha$ came from a multivariable fit of $\text{Re}_V = \tilde{R}_0 \text{Pr}^\alpha \text{Ra}^\eta$ to all data which gave $\alpha = -0.53 \pm 0.11$ and $\eta = 0.45 \pm 0.03$. The dashed line is $\text{Re}/\text{Pr}^\alpha$ from the GL model using $\alpha = -0.62$.

Figure 7. Processed shadowgraph images. Left: only the dark (relatively warm) structures near the bottom of the sample remain. Right: only the light (relatively cold) structures near the top of the sample remain. For both images a 94 mm square was used with a resolution of 5 pixels/mm. They are for $\text{Pr} = 0.88$ and $\text{Ra} = 2 \times 10^7$. Corresponding movies can be found in supplementary material submitted with this paper as Fig7a.mov and Fig7b.mov.

the range from 0.44 to 0.45, one finds that $\alpha$ varies from about -0.59 to -0.62. Although our $\alpha$ value falls below the range of the GL-model, its relatively large statistical uncertainty overlaps that range. The experimental result for $\eta$ is in good agreement with the model.

Figure 6 shows $\text{Re}_V/\text{Pr}^\alpha$ for the $\text{Re}_V$ data and for $\text{Re}_{GL}$ (dashed line), using $\alpha = -0.53$ for $\text{Re}_V$ and $\alpha = -0.62$ for $\text{Re}_{GL}$. We see that the measured $\text{Re}_V$ points largely collapse onto a single line as expected if $\text{Pr}^\alpha$ indeed represents the $\text{Pr}$-dependence of $\text{Re}_V$. $\text{Re}_V/\text{Pr}^\alpha$ agrees reasonably with $\text{Re}_{GL}/\text{Pr}^\alpha$ in both magnitude and $\text{Ra}$-dependence. We note again that there was no a priori reason to expect the agreement in magnitude because $\text{Re}_{GL}$ was based on $U$ for a $\Gamma = 1.00$ sample with $\text{Pr} = 5.4$ and not on a rms fluctuation velocity for large $\Gamma$ and $\text{Pr}$ less than one.
3.3. Re\textsubscript{V} near the top and the bottom of the sample

We examined whether the Ra-dependence of Re for the hot fluid near the bottom plate (relatively dark structures) differed from that of the cold fluid near the top plate (relatively bright structures). We did so by retaining only the dark or light structures in each image, using the image processing procedure described in Sec. 2.3. Examples of these two types of processed images are shown in Fig. 7, and movies are available as supplementary material. In Fig. 8 we show results for Re\textsubscript{V} for the original images (solid symbols), for the light structures (cold, open blue symbols), and for the dark structures (hot, open red symbols) in the reduced form of Re\textsubscript{V}/Ra\textsuperscript{0.45} as a function of Ra. Although there are small systematic differences between the three data sets, the general trends of the data with Ra are very similar. For the light structures a simultaneous fit of Re\textsubscript{V}/Pr\textsuperscript{α} = \tilde{R}_0Ra\textsuperscript{η} with α fixed at -0.53 to the data for all Pr yielded η = 0.46 ± 0.05, while the dark structures gave η = 0.44 ± 0.03. We conclude that the fluctuation velocities are similar near the top and bottom plates.

4. Summary

We reported experimental measurements of Reynolds numbers over the range 3 \times 10^5 \lesssim Ra \lesssim 2 \times 10^7 and for 0.18 \lesssim Pr \lesssim 0.88 for turbulent convection in a cylindrical sample of aspect ratio Γ = 10.9. The range of Pr was achieved by using pure gases and binary gas mixtures. We used correlation functions based on time series of shadowgraph images and the elliptic approximation to determine the Reynolds numbers.

The measured Reynolds number Re\textsubscript{U} was based on the mean flow velocity U. We found it to be very small, and attribute this to the averaging of the local flow velocity due to the random diffusion of large-scale flow cells in large-Γ samples.

The Reynolds number Re\textsubscript{V} based on the rms fluctuation velocity V had a size comparable to Re\textsubscript{U} predicted by the Grossmann-Lohse model (Grossmann & Lohse 2002) for a sample of unit aspect ratio where there is only a single LSC cell locked in place by the side walls. Both the Ra dependence and the Pr dependence found for Re\textsubscript{V} agreed within the experimental uncertainty with the GL prediction for Re\textsubscript{U} and Γ = 1.
Reynolds-number measurements for turbulent convection

Figure 9. Estimates of the Rayleigh number \( R_a_t \) at the transition to turbulence as a function of \( Pr \). Solid circle (red online): Experimental result for the onset of turbulence given in the supplementary material of Bosbach et al. (2012). Solid line: \( R_a_t = R_a(t_{coh}/L = 0.1) \). Dashed line: \( R_a_t = R_a(t_{coh}/L = 0.26) \). Dash-dotted line: \( R_a_t = R_a(t_{coh}/L = 1) \). The thin vertical dotted lines represent the values of \( Pr \) of the present work. The horizontal dotted line represents the smallest value \( R_a = 3 \times 10^5 \) of \( R_a \) for the data of this paper.

Direct numerical simulations (Verzicco & Camussi 1999) had suggested that plumes form in the thermal boundary layers adjacent to the top and bottom plates only when \( Pr \gtrsim 0.35 \). Contrary to this, we found plumes over the entire \( Pr \) range of our measurements, with no noticeable difference in their appearance or number (see Fig. 1a).

We are grateful to Stephan Weiss and Johannes Bosbach for their contributions during the early stages of this work. One of us (GA) is very grateful to Xiaozhou He for many illuminating discussions about the elliptic approximation and its use to determine Reynolds numbers. The work was supported by the U.S. National Science Foundation through Grant No. DMR11-58514.

5. Appendix: Onset of turbulent convection

At relatively small \( Ra \) the flow field of RBC forms patterns (Bodenschatz et al. 2000). With increasing \( Ra \) the patterns become time dependent; often this time dependence becomes chaotic as \( Ra \) increases further (Krishnamurti 1970) (for a detailed example, see for instance the supplementary material provided by Bosbach et al. (2012)). At some point, with increasing \( Ra \), the chaotic state undergoes a transition to turbulence. At least for some \( \Gamma \) and \( Pr \) this transition is discontinuous. For the case studied by Bosbach et al. (\( Pr = 0.73 \) and \( \Gamma = 10 \)) the transition occurred at \( R_a_t \simeq 3 \times 10^5 \). In another case studied in our laboratory (Sharf et al. 2011) it was also discontinuous and found to occur at \( R_a_t = 9 \times 10^6 \) for \( Pr = 232 \) and \( \Gamma = 1.00 \).

Whether or not the state immediately above this transition can be regarded as genuine turbulence is a matter of definition and of some debate. It has been suggested that a suitable criterion can be defined in terms a coherence length \( l_{coh} = 10\eta_K \) (Grossmann & Lohse 1993; Sugiyama et al. 2007) where

\[
\eta_K/L = Pr^{1/2} [Ra(Nu - 1)]^{-1/4} \sim Ra^{-\xi_{eff}} \tag{5.1}
\]

is the volume-averaged Kolmogorov length (see, for instance, Sugiyama et al. (2007)).
The coherence length is an estimate of the smallest size of coherent structures, or eddies, in the bulk. It has been suggested by Sugiyama et al. (2007) that the bulk is not fully turbulent until \( l_{coh}/L \) is as small as 0.1. This criterion is represented by the solid line in Fig. 9. We believe that it is too restrictive. The solid circle in the figure represents the measurement of the discontinuous transition from “patterns” to chaotic or turbulent behavior observed by Bosbach et al. (2012). Those authors noted that there were no further transitions and no noticeable change of the power laws describing certain features of the system as \( Ra \) increased. The dashed line in the figure which passes through the experimental point corresponds to \( l_{coh}/L = 0.26 \). We did not use any data falling below this line in our analysis.

REFERENCES


