Bifurcations in turbulent rotating Rayleigh-Bénard convection: A finite-size effect

Stephan Weiss and Guenter Ahlers
Department of Physics, University of California, Santa Barbara, CA 93106, USA
E-mail: guenter@physics.ucsb.edu

Abstract. In turbulent rotating Rayleigh-Bénard convection Ekman vortices extract hot or cold fluid from thermal boundary layers near the bottom or top plate and enhance the Nusselt number. It is known from experiments and direct numerical simulation on cylindrical samples with aspect ratio \( \Gamma \equiv D/L \) (\( D \) is the diameter and \( L \) the height) that the enhancement occurs only above a bifurcation point at a critical inverse Rossby number \( 1/Ro_c \), with \( 1/Ro_c \propto 1/\Gamma \). We present a Ginzburg-Landau like model that explains the existence of a bifurcation at finite \( 1/Ro_c \) as a finite-size effect. The model yields the proportionality between \( 1/Ro_c \) and \( 1/\Gamma \) and is consistent with several other measured or computed system properties. Here it is used to estimate the suppression of the heat-transport enhancement by the sidewall.

1. Introduction

It has long been known (Rossby, 1969) that heat transport, usually expressed in terms of the Nusselt number \( Nu \), in turbulent Rayleigh-Bénard convection can be enhanced significantly by rotating the sample at modest rates about a vertical axis while the Rayleigh number \( Ra \) and the Prandtl number \( Pr \) are held constant. It is well understood that this enhancement is caused by the Coriolis force, which spins thermal plumes emanating from the thermal boundary layers near the top and bottom plates into cyclonic vortex tubes known as Ekman vortices. These vortices extract additional warm (cold) fluid out of the bottom (top) boundary layer, thusly providing an additional heat-transport mechanism. Quantitative measurements and direct numerical simulations (DNS) of the dependence of this enhancement of \( Nu \) on \( Ra \) and \( Pr \) were reported recently by Zhong et al. (2009) and Zhong & Ahlers (2010). It is also well known that rapid rotation will depress \( Nu \) due to the Taylor-Proudman effect which suppresses flow parallel to the rotation axis. Both phenomena are illustrated well by Fig. 1a, which is adapted from Zhong & Ahlers (2010).

A remarkable feature of the data in Fig. 1a, first reported by Stevens et al. (2009), is that the \( Nu \) enhancement does not set in as soon as rotation starts, but rather occurs only after a critical value \( 1/Ro_c \) of \( 1/Ro \) has been reached. Such a bifurcation from one turbulent state to another initially came as a surprise because it was thought that the vigorous turbulent fluctuations would smooth out any potential transition between different states. However, bifurcations between turbulent states have been reported recently also in other systems, most recently for instance in a von Karman flow (de la Torre & Burguette, 2007;
Figure 1. (a): The Nusselt number $\text{Nu}(1/\text{Ro})$, normalized by $\text{Nu}(0)$ (i.e. without rotation), as a function of the inverse Rossby number $1/\text{Ro}$ on a logarithmic scale. Open circles: $Pr = 6.26$ and $Ra = 2.73 \times 10^8$. Solid circles: $Pr = 4.38$ and $Ra = 2.25 \times 10^8$. The small vertical dotted line shows the location of the critical inverse Rossby number $1/\text{Ro}_c \simeq 0.42$. Adapted from Zhong & Ahlers (2010). (b): An estimate of the ratio of the normalized excess heat transport $\delta\text{Nu}(\Gamma)$ of the finite system to $\delta\text{Nu}_b$ in the bulk (or for the infinite system) as a function of $1/\text{Ro}$ on a linear scale.

When they do occur, they are presumably associated with transitions in boundary conditions, boundary-layer structures, or internal large-scale structures. In the case considered here it is the appearance of a large-scale structure, the Ekman vortices, that is associated with the bifurcation (Weiss et al., 2010). However, the question remains why these vortices do not start to develop as soon as rotation starts, i.e. at $1/\text{Ro} = 0$. In the remainder of this paper we shall discuss this phenomenon in terms of a phenomenological model in the form of a Ginzburg-Landau equation (see for instance Cross & Greenside (2009)). This approach was presented first in Weiss et al. (2010). It reveals that the finite threshold is a finite-size (i.e. finite-$\Gamma$) effect, and that the bifurcation would cease to exist in the laterally infinite system because $\text{Nu}(1/\text{Ro})$ would start to increase with $1/\text{Ro}$ as soon as rotation starts at $1/\text{Ro} = 0$. It will be interesting to see whether a similar explanation applies to bifurcations in other turbulent systems, such as for instance the von Karman flows reported in de la Torre & Burguette (2007) and Cortet et al. (2010).

2. The Ginzburg-Landau model

We consider a Ginzburg-Landau like model

$$\dot{A} = (1/\text{Ro}^2) A - gA^3 + \xi_0^2 \nabla^2 A$$

for the local vortex-line density $A$ and assume that the normalized excess heat transport $\delta\text{Nu} = \text{Nu}(1/\text{Ro})/\text{Nu}(0) - 1$ is proportional to $\bar{A}$ where $\bar{A}$ is the spatial average of $A$ over the entire sample. We chose the coefficient of the linear term to be $(1/\text{Ro})^2$ because this yields $A = (1/\text{Ro})/\sqrt{g}$ for the time independent spatially uniform infinite system; i.e. in the absence of boundaries there is no bifurcation and the vortex density $A$, and thus $\delta\text{Nu}$, starts to rise linearly with $1/\text{Ro}$ (and thus with $\Omega$) as soon as rotation starts. The term $\xi_0^2 \nabla^2 A$ is the lowest-order term in a gradient expansion for a non-propagating state. The cubic nonlinearity yields a supercritical bifurcation as seen in the experiment.

Since our model is only semi-quantitative and phenomenological, we shall consider further only the simpler one-dimensional case (the two-dimensional equation was analyzed in detail in Ahlers et al. (1981) and yields very similar results). In that case standard stability analysis (Cross & Greenside, 2009) of Eq. 1 for an infinite system shows that the ground state $A = 0$ is stable to perturbations.
with wave number $k$ below a neutral curve given by $1/\text{Ro}_0(k) = \xi_0 k$. For the finite system we shall assume that $A$ must vanish at the sample wall because no vortices can be located at that point, i.e. $A(-\Gamma/2) = A(\Gamma/2) = 0$. This reasonable assumption has indeed been confirmed by DNS (Weiss et al., 2010; Stevens et al., 2011). This implies that the lowest mode that fits into the sample corresponds to $k_0 = \pi/\Gamma$. With the stability curve $1/\text{Ro}_0(k_0)$ this yields

$$1/\text{Ro}_c \equiv 1/\text{Ro}_0(k_0) = \pi \xi_0 / \Gamma$$

(2)

for the critical inverse Rossby number. Thus, consistent with the experiment, we find $1/\text{Ro}_c \propto 1/\Gamma$. Comparison with the experimental values of $1/\text{Ro}_c$ (Weiss et al., 2010) yields $\xi_0 = 0.12$. Further analysis (Weiss et al., 2010) reveals that there is a boundary layer of approximate thickness $\xi = \sqrt{2\xi_0 \text{Ro}}$ near the wall where the vortex density (and thus $\delta \text{Nu}$) is suppressed. Thus, there is a $\Gamma$-independent annulus of width proportional to $\text{Ro}$ just inside the sidewall where on average the heat flux is suppressed. The effect on the total heat flux will be relatively smaller for larger $\Gamma$ because the area inside of this annulus is larger. Thus, even if one makes the reasonable assumption that the vortex density in the sample interior is $\Gamma$-independent, one has to conclude that the total heat flux, and thus $\delta \text{Nu}$, is $\Gamma$-dependent. An estimate of the size of this effect (not too close to $1/\text{Ro}_c$) can be made by assuming that the vortex density in the bulk of the sample is $A_b = A_\infty$ where $A_\infty$ is the line density of the infinite system, and that an average line density $A_w$ prevails in the boundary layer. A reasonable value for $A_w$ might be $0.5 A_b$; a more accurate value could be obtained by integrating the solution of Eq. 1 over the boundary layer but is not necessary for the present approximate estimate. Based on simple considerations of the boundary and bulk areas, one then has

$$\delta \text{Nu}(\Gamma)/\delta \text{Nu}(\infty) = (1 - 2\xi/\Gamma)^2 + [4\xi/\Gamma - (2\xi/\Gamma)^2] A_w / A_b .$$

(3)

Results for this ratio (assuming $A_w/A_b = 0.5$) are shown in Fig. 1b for several values of $\Gamma$. One sees that, even for $1/\text{Ro}$ as large as five for instance, the global heat-transport enhancement is predicted to be significantly dependent upon $\Gamma$ due to the boundary influence. This is so even though it was assumed that the local heat-flux density in the sample interior was $\Gamma$-independent. It should be mentioned that Eq. 1 is not really expected to yield a good prediction of $\delta \text{Nu}$ for $1/\text{Ro}$ as large as five; nonetheless we believe that the results for the ratio shown in Fig. 1b give a good indication of the size of the heat-transport suppression due to the wall even at these large rotation rates.

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References


