Logarithmic temperature profiles in the bulk of turbulent Rayleigh–Bénard convection for a Prandtl number of 12.3

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We report measurements of logarithmic temperature profiles $\Theta(z, r) = A(r) \times \ln(z/L) + B(r)$ in the bulk of turbulent Rayleigh–Bénard convection (here $\Theta$ is a scaled and time-averaged local temperature in the fluid, $z$ is the vertical and $r$ the radial position, and $L$ is the sample height). Two samples had aspect ratios $\Gamma \equiv D/L = 1.00$ and 0.50 (where $D = 190$ mm is the diameter). The fluid was a fluorocarbon with a Prandtl number of $Pr = 12.3$. The measurements covered the Rayleigh-number range $2 \times 10^{10} \lesssim Ra \lesssim 2 \times 10^{11}$ for $\Gamma = 1.00$ and $3 \times 10^{11} \lesssim Ra \lesssim 2 \times 10^{12}$ for $\Gamma = 0.50$. In contradistinction to what had been found for $\Gamma = 0.50$ and $Pr = 0.78$ by Ahlers et al. (Phys. Rev. Lett., vol. 109, 2012, art. 114501; J. Fluid Mech., 2014, in press), the measurements revealed no $Ra$ dependence of the amplitude $A(r)$ of the logarithmic term. Within the experimental resolution, the amplitude was also found to be independent of $\Gamma$. It varied with $r$ in a manner consistent with the function $A(\xi) = A_1 / \sqrt{2\xi - \xi^2}$, where $\xi \equiv (R - r)/R$ with $R = D/2$ and $A_1 \approx 0.0016$. The results for $A(r)$ are smaller than those obtained from experiments and direct numerical simulations (Ahlers et al., Phys. Rev. Lett., vol. 109, 2012, art. 114501) at similar values of $Ra$ for $Pr = 0.7$ and $\Gamma = \frac{1}{2}$ by a factor that depended slightly upon $Ra$ but was close to 2.

Key words: Bénard convection, convection

1. Introduction

Rayleigh–Bénard convection (RBC) (for various reviews, see Kadanoff 2001; Ahlers 2009; Ahlers, Grossmann & Lohse 2009; Lohse & Xia 2010; Chillà & Schumacher 2012) occurs when a fluid contained between two horizontal parallel plates is heated from below. The applied temperature difference $\Delta T = T_b - T_t$ (where $T_b$ and $T_t$ are the bottom and top temperatures, respectively) is usually expressed in dimensionless form via the Rayleigh number $Ra$, defined below by (2.1). When $Ra$ becomes sufficiently large, the convective flow becomes turbulent.

When $Ra$ is large but not too large, turbulent RBC is well understood to contain thin thermal (see e.g. Lui & Xia 1998) and viscous (see e.g. Qiu & Xia 1998a; Lam et al. 2002) boundary layers (BLs) near the top and bottom plates. This state is now referred to as ‘classical’ RBC. Approximately half of $\Delta T$ is found across each of

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the thermal BLs. The remainder of the fluid, known as the ‘bulk’, has long been
considered isothermal in the time average (Malkus 1954; Priestley 1954, 1959; Spiegel
1971), albeit with a vigorously fluctuating temperature. More recent investigations (see,
for instance, Tilgner, Belmonte & Libchaber 1993; Brown & Ahlers 2007b) revealed
that the bulk does contain some temperature variations, but these were thought to
consist mostly of constant gradients caused by plumes emanating from the thermal
BLs and propagating through the bulk. For a cylindrical sample, these gradients were
found to depend on the radial position (Brown & Ahlers 2007b) as well as on the ratio
of the kinematic viscosity to the thermal diffusivity, known as the Prandtl number \( Pr \);
see (2.2) below.

Recently it was found by experiment and confirmed by direct numerical simulation
(DNS) that the temperature field in the bulk of a cylindrical sample of aspect ratio
\( \Gamma \equiv D/L = 0.50 \) (where \( D \) is the diameter and \( L \) the height of the cylinder) is
actually far more interesting (Ahlers et al. 2012; Ahlers, Bodenschatz & He 2014).
Measurements of the local time-averaged and scaled dimensionless temperature
\( \Theta(z, r) \equiv [(T(z, r, t) \langle \rangle - T_m)/\Delta T \) (here \( T_m \equiv (T_b + T_t)/2 \) and \( \langle \rangle \) indicates the average
over the time \( t \) for a statistically stationary process) could, within the experimental
uncertainty, be represented by

\[
\Theta(z, r) = A(r) \ln(z/L) + B(r)
\]

near the bottom plate and by

\[
\Theta(z, r) = A'(r) \ln(1 - z/L) + B'(r)
\]

near the top plate. Here \( z \) is the distance from the bottom plate and \( r \) is the horizontal
distance from the vertical centreline. These logarithmic layers extended a significant
distance away from the BLs into the bulk of the fluid. For that work, the Prandtl
number was 0.78. The measurements yielded values of \( A \) that varied slightly with
\( Ra \), approximately as \( A \propto Ra^{-0.12} \), and depended strongly on \( r \), approximately as \( A = A_0(1 - r/R)^{-0.67} \) where \( R = D/2 \).

The discovery of the logarithmic temperature field in RBC came as a complete
surprise. Detailed understanding of it will be enhanced significantly by further
measurements which elucidate the dependence of the coefficients in (1.1) and (1.2)
on the free parameters of this system. These parameters are \( \Gamma \) and \( Pr \). One might
expect that \( A \) and \( A' \) should be independent of \( \Gamma \), since the logarithmic contribution to
the profile might be expected to depend only on the distance from the bottom or the
top plate and not on the overall height of the system; but this expectation is based on
a preconceived idea about the origin of the profiles, and an experimental verification
of this surmise is of utmost importance for the development of a firm theoretical
framework. Equally important is to learn something about the dependence of the coefficients \( A \) and \( B \) of the profile on the Prandtl number, since this dependence should
be contained in a theoretical model that may eventually be developed. Experimentally
this is a difficult task, since most large-\( Pr \) fluids do not allow access to very large
\( Ra \) with samples of laboratory-sized heights, and large values of \( Ra \) are necessary
to assure that the BLs are thin enough for the log profile outside of them in the
bulk to exist over a significant range of \( z \). From this viewpoint, the fluid used in
the present investigation, a fluorocarbon known as FC72 (see § 3.1), is particularly
suitable as it allows access to \( Ra \) up to \( 2 \times 10^{12} \) under nearly Boussinesq conditions
with the reasonable height \( L \simeq 19 \text{ cm} \) and for \( Pr = 12.3 \). For comparison, a sample
of ethanol (which has a similar Prandtl number, \( Pr \simeq 15 \) with the same height and
\( \Delta T \) would give \( Ra \approx 2 \times 10^{10} \), which is two orders of magnitude smaller and would have correspondingly thicker BLs that might tend to hide the logarithmic layers.

Many aspects of the results for \( Pr \approx 0.7 \) or 0.8 (Ahlers et al. 2012, 2014) are similar to those for various shear flows (for recent reviews see, for instance, Pope 2000; Marusic et al. 2010; Smits, McKeon & Marusic 2011), where the logarithmic dependence for the time-averaged downstream (‘streamwise’) velocity, known as the ‘law of the wall’, was first enunciated by Prandtl (1925, 1932) and von Kármán (1930) (see also Millikan 1938) and has been studied intensively ever since (for a detailed discussion of the similarities and differences between the logarithmic dependences in RBC and shear flows, see § 4.3.2 of Ahlers et al. 2014). A derivation of the logarithmic temperature profile from a two-sublayer mean-field model for RBC was presented in § 4.3.1 of Ahlers et al. (2014).

In this paper we present new measurements of \( \Theta(z, r) \) using a fluid with a Prandtl number that is approximately a factor of 15 larger than that of the previous investigation. Measurements were made for two samples with the same diameter, \( D = 190 \text{ mm} \), one with \( \Gamma = 1.00 \) and the other with \( \Gamma = 0.50 \). Also for these cases, the measurements were consistent with the existence of a logarithmic layer in the bulk where the temperature field varied logarithmically with the distance from the plates. The amplitude \( A \) decreased as the distance from the sidewall increased. We found no dependence of \( A \) on \( \Gamma \). In contradistinction to the results for \( Pr \approx 0.8 \), we also found no dependence of \( A \) on \( Ra \).

The paper is organized as follows. In § 2 we define parameters relevant to this system. Section 3 gives experimental details, including a discussion of the fluid that was used (§ 3.1), of the experimental facility and convection cell (§ 3.2) and of the thermistors employed for the local temperature measurements (§ 3.3). The results are presented in § 4; they include measurements of the sample centre temperature (non-Boussinesq effects, § 4.1) and the Nusselt number (§ 4.2). Both these results were included primarily to show that our main measurements, reported in § 4.3, pertain to the Boussinesq system, and that any uncertainties of the fluid properties do not unduly influence the values of \( Ra \) derived from the temperature measurements. The major part of the results is given in § 4.3, where the logarithmic temperature profiles in the bulk of the sample are presented. Section 4 concludes with measurements of the temperature gradient near the sample centre (§ 4.4), which for \( \Gamma = 1.00 \) turns out to be stabilizing. The body of the paper concludes with a brief summary (§ 5) and is then followed by an Appendix which gives the axial and radial locations of all thermistors used in this work.

2. Relevant parameters

For a given sample geometry, the state of the system depends on two dimensionless variables: the Rayleigh number \( Ra \) (a dimensionless form of \( \Delta T \)) and the Prandtl number \( Pr \) (the ratio of the viscous to the thermal diffusivity). They are given by

\[
Ra = \frac{g \alpha \Delta T L^3}{\kappa \nu} \quad (2.1)
\]

and

\[
Pr = \frac{\nu}{\kappa}. \quad (2.2)
\]

Here \( g, \alpha, \kappa \) and \( \nu \) denote the gravitational acceleration, the isobaric thermal expansion coefficient, the thermal diffusivity and the kinematic viscosity, respectively.
For the measurements reported here, all fluid properties are evaluated at the mean temperature \( T_m \equiv (T_b + T_t)/2 \).

The vertical heat transport from the bottom plate to the top plate is expressed in dimensionless form by the Nusselt number

\[
Nu = \frac{\lambda_{\text{eff}}}{\lambda},
\]

where \( \lambda \) is the conductivity of the quiescent fluid and \( \lambda_{\text{eff}} \) is the effective conductivity in the presence of convection, given by

\[
\lambda_{\text{eff}} = \frac{Q}{A} \frac{L}{\Delta T},
\]

with \( Q \) being the heat flux and \( A \) the cross-sectional area of the cell.

For samples in the shape of right circular cylinders like those used here, a further parameter defining the geometry is needed, namely the aspect ratio \( \Gamma \equiv D/L \).

In the axial direction we shall use \( z/L \) in the lower half and \( 1 - z/L \) in the upper half of the sample as the relevant dimensionless coordinate. Here \( z \) is the distance from the bottom plate of the sample. For the radial position we shall use the parameter

\[
\xi \equiv (R - r)/R,
\]

where \( r \) is the radial distance from the centreline and \( R = D/2 \).

For real fluids, the sample centre temperature \( T_c \) will differ from the mean temperature \( T_m \) due to non-Oberbeck–Boussinesq effects. A measure of the size of these effects is the parameter

\[
\phi \equiv (T_c - T_m)/\Delta T.
\]

### 3. Apparatus and procedures

#### 3.1. The fluid

The experiments presented here were conducted using the 3M proprietary fluorocarbon Fluorinert™ FC72 at a mean temperature \( T_m = 25.00^\circ \text{C} \), where \( Pr = 12.3 \). The manufacturer cites an average molecular weight of 340 g mol\(^{-1}\), suggesting a composition of nearly 100 % of \( \text{C}_6\text{F}_{14} \) (MW = 338.0).

The vapour pressure as a function of temperature is given by Crowder et al. (1967). There the normal boiling point of \( n\text{-C}_6\text{F}_{14} \) is given as 330.3 K (57.1 °C). The manufacturer cites the ‘typical’ boiling point of FC72 as 56 °C, not too far from the normal boiling point of \( n\text{-C}_6\text{F}_{14} \). The relevant fluid properties are given at a temperature of 25 °C by the manufacturer, most of them to only two significant digits. Assuming that the uncertainties are \( \pm 0.5 \) in the least significant digit, and making the pessimistic assumption that all error contributions to \( Ra \) add, we estimate that the maximum possible systematic error of \( Ra \) is 9 %; but this may well be an overestimate because some error cancellation will most likely take place. Similarly, the maximum possible systematic error of \( Nu \) due to systematic errors of \( \lambda \) is 1 %. We find the corresponding maximum possible uncertainty for \( Nu/Ra^0.3 \) to be approximately 4 %.

For the \( \Gamma = 0.50 \) sample, measurements up to \( Ra \simeq 2 \times 10^{12} \) can be made with \( \Delta T \leq 30 \text{ K} \). Approximately 12 l (20 kg) of liquid was required. The cost of approximately US $3000 made it impractical to study an even larger sample that could have reached even higher values of \( Ra \). Note that at constant \( \Gamma \), the largest \( Ra \), corresponding to a fixed largest \( \Delta T \), is proportional to the volume (and thus cost) of the fluid.
3.2. The apparatus

We used a modified version of the apparatus described in detail by Zhong & Ahlers (2010). Cylindrical sample cells, one with \( L = D = 190 \) mm \((\Gamma = 1.00)\) and one with \( D = 190 \) and \( L = 380 \) mm \((\Gamma = 0.50)\), were used. The samples were confined by thick copper plates from below and above with six rods connecting two aluminium brackets, one each below the bottom plate and above the top plate. The sidewall had a thickness of 6.35 mm and was made of Lexan\textsuperscript{TM} (polycarbonate resin). A groove in the top and bottom plates directly below or above the sidewall contained an ‘O’-ring which sealed the wall to the plates. Two thin Teflon\textsuperscript{TM} (polyimide) capillaries, one near the top and the other near the bottom of the cell, penetrated the sidewall and were used to fill the sample. The other ends of the capillaries terminated in small reservoirs located at a vertical position well above the sample. The bottom plate was temperature-controlled in a feedback loop by a film heater from below with milli-Kelvin stability, and the top-plate temperature was controlled by circulating water coming from a circulator with 0.01 K stability. The sample was partially degassed by heating the bottom plate of the filled cell to approximately 50\( ^\circ \)C for many hours before any sidewall shields were installed. The unregulated top plate was then at approximately 40\( ^\circ \)C. Any air that had thereafter accumulated at the cell top was eliminated. During the measurements, the bottom-plate temperature always remained below 40\( ^\circ \)C, and no further air bubbles formed.

The sample cell was levelled within less than \( 10^{-4} \) rad. It sat on an equally levelled rotating table (see Zhong & Ahlers 2010), which, however, was not rotated for the present work.

3.3. The thermistors

Measurements of local temperatures in the bulk of turbulent convection are challenging because many fluids are not perfect insulators and bulky liquid-proof insulation on the leads and thermistor bead is undesirable. An advantage of the fluorocarbons (which they share also with gases and some silicone oils such as polydimethylsiloxane, for instance) is that they are near-perfect insulators with a typical volume resistivity near \( 10^{15} \ \Omega \) cm, allowing the bare thermistors and leads to be inserted directly without additional insulation.

In the bulk of the fluid we used the two thermistor types shown in figure 1(c). They were inserted to various depths into the sample interior. They differed in size and thus in thermal response time. While the time response is important for fluctuation measurements, for the present measurements of time-averaged temperatures the two types served equally well and gave equivalent results.

The larger thermistors, to be referred to as \( T_1 \), were Honeywell model 112-104KAJ-B01 and were used in the \( \Gamma = 1.00 \) sample. They were glass-encapsulated, and the glass capsule had an outside minor diameter of 1.1 mm and a length of approximately 1.8 mm. The smaller thermistors were Honeywell model 111-104HAK-H01, to be called \( T_2 \), and were used in the \( \Gamma = 0.50 \) sample. They had diameters of 0.36 mm and were also glass-encapsulated. Thermistors \( T_2 \) had platinum-alloy leads of 0.02 mm diameter.

Forty \( T_1 \) thermistors were inserted to various depths into the interior of the \( \Gamma = 1.00 \) sample, and 36 \( T_2 \) thermistors were inserted into the \( \Gamma = 0.50 \) sample. Their leads passed, one each, through two 0.13 mm diameter holes parallel to the axis of 0.8 mm diameter ceramic rods (Omega ceramic thermocouple insulators of type TRA-005132), as illustrated in figure 1(c) for \( T_2 \). The rods were placed in holes of
Figure 1. (Colour online) (a,b) Time series (with arbitrary time origin) of the resistances of two thermistors at a nearly constant temperature, one each at the centre of the two convection samples: (a) thermistor $V_{0.5}$ (see table 2) of type $T_1$; (b) thermistor $V_{0.7}$ (see table 2) of type $T_2$. For both cases the mean temperature was $T_m = 25.00^{\circ}C$ and $\Delta T = 0.10$ K, corresponding to $Ra \simeq 8 \times 10^8$ for $\Gamma = 1.00$ and $Ra = 7 \times 10^9$ for $\Gamma = 0.50$. (c) Photographs of the two thermistor types (the scale divisions are 1 mm apart).

0.9 mm diameter in the Lexan sidewall and sealed to the sidewall with epoxy, as can be seen in figure 2.

In figure 1(a,b) we illustrate the stability and noise level of the two thermistors while inserted in FC72 at an essentially constant temperature. For both cases any temperature drift was at the milli-Kelvin level or less over a $10^4$ s period, and the root-mean-square resistance fluctuations were equivalent to temperature fluctuations of the order of 1 mK.

Larger thermistors (Honeywell type 121-503JAJ-Q01) were calibrated in a separate facility with a precision of 1 mK against a Hart Scientific Model 5626 platinum resistance thermometer, which in turn had been calibrated against various fixed points on the ITS-90 temperature scale by the Hart Scientific Division of Fluke Corporation. These thermistors were inserted into deep horizontal holes in the top and bottom plates and provided measurements of $T_b$ and $T_t$.

The tables in the Appendix show the axial and radial positions of all thermistors located in the bulk. A photograph of the sidewall for the $\Gamma = 1.00$ sample (laid on its side and viewed from below) is shown in figure 2. The eight thermistor columns listed in the tables are indicated in figure 2 as $V_i$ for $i = 0, \ldots, 7$.

For the $\Gamma = 1.00$ sample, an additional eight type-$T_1$ thermistors were embedded in the sidewall at the middle height $z/L = 0.50$. They were used to measure the centre temperature $T_c$. For the $\Gamma = 0.50$ sample there were three sets of eight $T_2$ thermistors equally spaced in horizontal circles at heights $z/L = 0.25, 0.50$ and 0.75. These thermistors penetrated the sidewall and extended into the fluid by 6 mm.

All thermistors in the bulk were calibrated against the top- and bottom-plate thermistors after they had been installed in the sidewall and the apparatus assembled.
As described by Zhong & Ahlers (2010), the plate temperatures were set to a desired value of $T_m$, and a small temperature difference $\Delta T \simeq 0.10 \, K$ ($Ra \simeq 10^9$) was applied to ensure equilibration of the system in a reasonable time (half a day or so). Measurements over several additional hours were then averaged, and the thermometers were assumed to be at $T_m$.

The resistances of all thermometers were usually measured every 11 s. Typically, the data from the first 6 h were discarded to eliminate transient effects. Measurements during at least the following 6 h were averaged to obtain the time-averaged temperatures. Some runs (e.g. 1310071) were extended to 3 days in order to ensure that no slow temperature transients remained and to obtain better statistics.

4. Results

4.1. Non-Oberbeck–Boussinesq effects

Many theoretical considerations are based on the Oberbeck–Boussinesq (OB) approximation (Oberbeck 1879; Boussinesq 1903), which assumes that the fluid properties, except for the density where it provides the buoyancy force, do not vary over the range of temperatures encountered in the samples. However, the properties of real fluids will deviate from this approximation to a greater or lesser degree, depending on the fluid and the applied temperature difference. It is important to have some indication of the size of this non-OB effect in order to be sure that the remaining data in this paper are representative of the Boussinesq system.

Non-OB effects will influence bulk properties such as $Nu$. They will also lead to a difference between the temperature drops across the top and bottom BLs. Thus the centre temperature $T_c$ will differ from the mean temperature $T_m$, and $T_c - T_m$ is a relatively sensitive measure of the importance of non-OB effects.

In figure 3 we show measurements of $T_c - T_m$ as a function of $\Delta T$. The red circles (respectively, purple diamonds) are values of $T_c$ obtained by averaging the eight...
Figure 3. (Colour online) The shift $T_c - T_m$, in kelvins, of the centre temperature $T_c$ relative to the mean temperature $T_m$ as a function of $\Delta T$. The solid black squares represent measurements by Ahlers et al. (2006) for a sample with $\Gamma = 1.00$ using thermistors embedded in the sidewall in the horizontal midplane and using water as the fluid. The other symbols indicate results from the present samples: red circles show the average temperature of the eight thermistors embedded in the sidewall in the horizontal midplane for $\Gamma = 1.00$, and purple diamonds represent the average temperature of the thermistors immersed 6 mm into the sample through the sidewall in the horizontal midplane for $\Gamma = 0.50$.

temperatures of the sidewall thermometers at $z = L/2$ for the $\Gamma = 1.00$ (respectively, $\Gamma = 0.50$) sample. The results for the two aspect ratios are very close to each other. This is consistent with the expectation that non-OB effects should depend only on the temperature differences across the BLs, and not on sample geometry or Rayleigh numbers. Previously (see figures 4 and 5 of Ahlers et al. 2006), it had been shown that the same values for $T_c - T_m$ were obtained at the same $\Delta T$ for $\Gamma = 1$ samples of water with different physical sizes (i.e. different values of $L$ and $D$ while $L = D$).

The results for FC72 are compared with the previous measurements by Ahlers et al. (2006) for a $\Gamma = 1.00$ sample of water with $Pr = 4.3$, shown as black solid squares in figure 3. One sees that non-OB effects for water are considerably larger than for FC72. Extensive measurements of $Nu$ and $Re$ for water showed that these global properties were not influenced measurably until $\Delta T$ reached values near 16–20 K. Thus, we believe that the global properties of FC72 will be very close to those of the OB system for $\Delta T$ up to 25 K or more. The bulk temperature field will, of course, be shifted, but we expect this to be a nearly uniform shift by $T_c - T_m$ or, in dimensionless form, by $\phi$. Unfortunately, the temperature dependences of the relevant fluid properties of FC72 are not known to us; thus it is difficult to know the reason for the lesser non-OB effect for FC72 compared to that for water; most likely, however, it would be found in a smaller temperature dependence of the viscosity.

We note that even at the largest $\Delta T \simeq 28$ K used in this work, $\phi$ only reached the small value of $\phi \simeq 0.015$. This is an order of magnitude smaller than the largest values of $\phi$ encountered in the work of Urban et al. (2012). Thus, as discussed also by He et al. (2013), the concerns expressed by Urban et al. (2012) are irrelevant to the present work.
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Figure 4. (Colour online) Plots of (a) the Nusselt number $Nu$ and (b) the reduced Nusselt number $Nu/Ra^{0.30}$ as functions of the Rayleigh number $Ra$ on logarithmic scales. Open black squares and red triangles represent uncorrected results for $\Gamma = 1.00$ and 0.50, respectively; solid black circles and red diamonds represent $Nu$ after a correction for the influence of the sidewall using model 2 of Ahlers (2000) for $\Gamma = 1.00$ and 0.50, respectively; open black circles and red diamonds represent $Nu$ after sidewall correction using the model of Roche et al. (2001) for $\Gamma = 1.00$ and 0.50, respectively. In (b), the solid and dash–dot lines are fits of the power law $Nu = Nu_0 Ra^{\gamma_{\text{eff}}}$ to the data, while the dashed line indicates the model of Grossmann & Lohse (2000, 2001) for $\Gamma = 1$ and $Pr = 12.3$ with parameters from Stevens et al. (2013).

4.2. The Nusselt number

As a check on the overall validity of our measurements with FC72, we took heat-transport measurements and compare them in this section with results expected from the Grossmann–Lohse model (Stevens et al. 2013), which, for this purpose, serves as an ‘interpolation formula’ between other experimental data that were used to determine the parameter values.

Results for the uncorrected Nusselt number are shown in figure 4 for $\Gamma = 1.00$ as black open squares and for $\Gamma = 0.50$ as red open triangles. They were obtained using $\Delta T \lesssim 28.8 \degree C$. In § 4.1 we estimated that corrections for non-OB effects are negligible for this fluid and these temperature differences. Corrections for the finite conductivity of the top and bottom plates (Brown et al. 2005) are likewise negligible, because the conductivity of the fluid is relatively small ($0.057 \text{ W m}^{-1} \text{ K}^{-1}$ at $25 \degree C$, compared, for instance, to $0.61 \text{ W m}^{-1} \text{ K}^{-1}$ for water at the same temperature). However, a correction for the nonlinear sidewall effect (Ahlers 2000; Roche et al. 2001) must be made. It was applied using model 2 of Ahlers (2000) and the model of Roche et al. (2001) with a free parameter of that model fixed at $A = 0.8$. For $\Gamma = 1.00$, this yielded the black solid and open circles, respectively, in figure 4; for $\Gamma = 0.50$, it gave the red solid and open diamonds, respectively.

It is well known from both theory (Grossmann & Lohse 2000, 2001) and experiment (Xu, Bajaj & Ahlers 2000) that the dependence of $Nu$ upon $Ra$ cannot be represented by a simple power law, and that a fit of

$$Nu = Nu_0 Ra^{\gamma_{\text{eff}}}$$

(4.1)

to $Nu(Ra)$ over a limited range of $Ra$ will yield an effective exponent $\gamma_{\text{eff}}$ which varies slowly with $Ra$. The black solid and dash–dot lines in figure 4(b) represent such fits...
to the solid and open circles (both for $\Gamma = 1.00$), respectively, which gave $Nu_0 = 0.087$, $\gamma_{\text{eff}} = 0.318 \pm 0.001$ and $Nu_0 = 0.089$, $\gamma_{\text{eff}} = 0.317 \pm 0.001$, respectively. The difference between the results from the two models for the sidewall correction reflects, we believe, the uncertainty of this correction. Within experimental uncertainty, both yield the same effective exponent.

The dashed line in figure 4(b) is the model of Grossmann & Lohse (2000, 2001) (GL) based on the coefficients for $\Gamma = 1$ from Stevens et al. (2013) and $Pr = 12.3$. The difference between the model prediction and the sidewall-corrected data is well within combined possible systematic errors of the GL prediction and errors due to uncertainties in the fluid properties, as discussed in §3.1. More important than the absolute value of $Nu$ is that a fit of (4.1) to the model prediction over the experimental $Ra$-range for $\Gamma = 1.00$ gives $\gamma^{\text{GL}}_{\text{eff}} = 0.315$, which is remarkably close to the experimental result. We are not aware of previous measurements for the Prandtl number and $Ra$-range of our data with which we could compare our results.

The results for $\Gamma = 0.50$ are very close to the extrapolation of the power-law fit to the data for $\Gamma = 1.00$ (the solid black line in figure 4b). This near-absence of an aspect-ratio dependence of $Nu$ is consistent with previous findings (Xu et al. 2000; Nikolaenko et al. 2005; Sun et al. 2005). The line passing through the solid red diamonds in figure 4(b) is a fit of (4.1) to the $\Gamma = 0.50$ data. It yields the exponent $\gamma_{\text{eff}} = 0.321 \pm 0.001$. A similar fit to the GL model over the $Ra$-range of the $\Gamma = 0.50$ data gave $\gamma^{\text{GL}}_{\text{eff}} = 0.320$, which is in excellent agreement with the experimental value for $\Gamma = 0.50$.

We note that, as expected, all effective exponents found in the present work are less than $\frac{1}{3}$, which is the asymptotic value of the GL model for the classical state as $Ra$ is extrapolated to infinity.

The consistency of our $Nu(Ra)$ with values in the literature as reflected in the GL model suggests that the fluid properties used to determine $Nu$ and $Ra$ will yield reasonable values of $Ra$ for the measurements in the next section.

### 4.3.1. Axial and radial dependence

In figure 5(a,b) we plot $\Theta$ for $\Gamma = 1.00$ as a function of $z/L$ for all five values of $\xi$ and for $Ra = 2.07 \times 10^{11}$ ($\Delta T = 24.8$ K). The results for the smallest $\xi = 0.020$ (blue squares) are much higher than any of the others, and are shown separately in figure 5(a). It is likely that the corresponding thermistors are located in or very close to the viscous BL adjacent to the sidewall (Qiu & Xia 1998b; Lam et al. 2002) rather than in the bulk. It is interesting that (but not understood why) a log profile still prevails in or very near the BL, but with an amplitude that is larger than would be expected from extrapolation of the bulk data (see also figure 10 below). The log profile breaks down, however, very near the bottom plate ($z/L = 0.020$, the open blue square in figure 5a). There the probe is located not only very close to or in the BL at the sidewall but also very close to the viscous BL that forms on the bottom plate. When the sidewall meets the bottom plate, the boundary-layer structure presumably becomes complex and thicker than near the plate centre (Lui & Xia 1998; Qiu & Xia 1998a). Although one could argue that the thermal BL should not have a significant influence because the estimate of its thickness, $\lambda_{\text{th}}/L = 1/(2Nu) = 0.0014$, is much less than the probe location $z/L = 0.020$, it is unclear whether these arguments apply when both the sidewall and the bottom plate come into play simultaneously.

The asterisks in figure 5(a) are the results for $Ra = 2.83 \times 10^{10}$ ($\Delta T = 3.38$ K) and $\xi = 0.020$. Well away from the bottom plate, at $z/L = 0.100$ and 0.210, these data are
Figure 5. (Colour online) Time-averaged temperatures $\Theta(z/L, \xi) = [\langle T(z, \xi) \rangle - T_m]/\Delta T$ as a function of the vertical position $z/L$ on a logarithmic scale. Panels (a) and (b) show results for $\Gamma = 1$; all data except the asterisks are for $Ra = 2.07 \times 10^{11}$ (run 1310071, $\Delta T = 24.8$ K). Symbols: blue squares, $\xi = 0.020$; red circles, $\xi = 0.062$; purple diamonds, $\xi = 0.210$; black downward-pointing triangles, $\xi = 1.00$; blue asterisks, $Ra = 2.83 \times 10^{10}$ and $\xi = 0.020$ (run 1310061). The lines are fits of (1.1) to the data with $z/L \leq 0.1$, except for $\xi = 0.020$ where the point at $z/L = 0.020$ was omitted but the one at $z/L = 0.210$ was included. The dash–double-dot line in (a) is shown also in (b) to provide an indication of the different vertical scales. The horizontal error bars on the diamonds ($\xi = 0.210$) correspond to an uncertainty of $\pm 0.5$ mm of the vertical position; approximately the same error bars apply to the data sets at all other $\xi$ values. Panel (c) shows results for $\Gamma = 0.50$ and $Ra = 1.3 \times 10^{12}$ (run 1404151, $\Delta T = 20.0$ K). Symbols: solid red circles, $\xi = 0.063$; open purple diamonds, $\xi = 0.116$; open downward-pointing black triangles, $\xi = 0.325$; solid upward-pointing green triangles, $\xi = 1.00$.

slightly displaced vertically relative to the results at larger $Ra$, but reveal the same slope (value of $A$). Closer to the plate, they show a larger deviation from the log profile, as might be expected at the smaller value of $Ra$ where all BLs are thicker.

The other four data sets, shown in figure 5(b), are expected to be in the bulk, well removed from the sidewall BL. In view of the relatively small size of the sample (compared to that of Ahlers et al. 2012, 2014) the uncertainty of the scaled vertical position $z/L$ will play a more significant role. Although we do not have a precise value, we estimate that a typical probable error is approximately $\pm 0.5$ mm, although it could be a bit larger for the thermistors located far away from the sidewall. This uncertainty is indicated on the set of data for $\xi = 0.210$ (purple diamonds). Within these uncertainties, the data for $z/L \leq 0.1$ fall on straight lines and can be fitted well with (1.1). As was found by Ahlers et al. (2012, 2014) for $Pr \simeq 0.8$ and $\Gamma = 0.50$, the data show that, also for our $Pr = 12.3$ and $\Gamma = 1.00$ case, the temperature field in the
bulk is consistent with a logarithmic dependence on the vertical position, and that the amplitude of the logarithmic term decreases as the sample centreline is approached.

Figure 5(c) displays results similar to those in figure 5(b), but for \( \Gamma = 0.50 \) and \( Ra = 1.3 \times 10^{12} \). Again the log profiles are evident. The solid circles (red online) and upward-pointing triangles (green online) are at nearly the same radial positions \( \xi \) for the two \( \Gamma \) values and can be compared directly. One sees that the amplitudes of the logarithmic term (slopes of the lines) are roughly similar. A more detailed comparison will be given below in figure 10.

The vertical temperature profile for \( \Gamma = 1.00 \) at the radial position \( \xi = 0.062 \) for \( Ra = 2.07 \times 10^{11} \) is shown in more detail in figure 6(a) on a linear scale and in (b) on a horizontal logarithmic scale. The data in figure 6(b) are consistent with a logarithmic dependence of \( \Theta \) on \( z/L \), except right at \( z/L = 0.5 \), where small deviations are seen.

Fitting (1.1) to the four points with \( z/L < 0.5 \) gave \( A = -(4.50 \pm 0.11) \times 10^{-3} \) and \( B = 0.0112 \pm 0.0003 \), while fitting (1.2) to the three points with \( 1 - z/L < 0.5 \) yielded \( A' = 4.39 \times 10^{-3} \) and \( B' = 0.0159 \) (based on only three points it does not seem justified to establish probable errors for \( A' \) and \( B' \), but these should be of the same order as those for \( A \) and \( B \)). We see that the results are consistent with \( A = -A' \), as expected for a system that is symmetric about its horizontal midplane and as was found also by Ahlers et al. (2014) for \( Pr \approx 0.8 \) in the classical state.

In figure 6(c,d), results similar to those in panels (a) and (b) are shown, but for \( \Gamma = 0.50 \) and \( Ra = 1.33 \times 10^{12} \). Again the log profile is well supported by the data in (d), but fits of (1.1) and (1.2) to the data with \( z/L < 0.1 \) or \( 1 - z/L < 0.1 \) show that deviations from this profile occur earlier than they do for \( \Gamma = 1.00 \). These fits yielded \( A = (-3.86 \pm 0.03) \times 10^{-3} \), \( B = (1.13 \pm 0.01) \times 10^{-2} \), \( A' = 3.80 \times 10^{-3} \) and \( B' = 1.32 \times 10^{-2} \). Thus, also for this case we regard the data to be consistent with \( A = -A' \).

To further demonstrate the consistency of the data with a logarithmic profile, we show in figure 7(a) the deviations of \( \Theta \) from the fits (solid lines) in figure 6. Here the solid symbols are for \( \Gamma = 1.00 \), and the open symbols represent the measurements for \( \Gamma = 0.50 \). The red circles are for deviations from the fits for \( z/L < 0.5 \) and the blue diamonds for \( 1 - z/L < 0.5 \). From this figure one sees more clearly that the data for \( \Gamma = 1 \) (solid symbols) follow the logarithmic profile up to \( z/L \simeq 0.3 \) or so, while the \( \Gamma = 0.50 \) data (open symbols) deviate somewhat sooner, say for \( z/L \gtrsim 0.15 \). In the logarithmic range there is no evidence of systematic deviations from the fits. The root-mean-square deviation from zero of all the data for \( \Theta - \Theta_{fit} \) in the logarithmic range is approximately \( 1 \times 10^{-4} \). In physical units this corresponds to approximately 2 mK, and is well within possible experimental errors.

It was conjectured by Ahlers et al. (2014) that the log profiles in RBC are caused by plumes emanating from the thermal BLs adjacent to the plates, in a broad sense similar to the generation of the log profiles in the averaged streamwise velocity in shear flows by coherent eddies evolving from the viscous sublayer near the wall. Based on this conjecture and on the results derived from a two-sublayer mean-field model for the temperature profile generated by the plumes, it was argued that a suitable length scale for the regime diagram of turbulent convection would be the thermal boundary-layer thickness \( \lambda_{th} \equiv L/(2Nu) \), which is believed to determine the thickness of plumes that are emitted from the BL, in analogy to the viscous sublayer thickness \( \delta_v \) in shear flow, which controls the original size of coherent eddies. For the case of figure 6 we have \( Nu = 343 \) and thus \( \lambda_{th} = 0.28 \text{ mm} \) \( (\lambda_{th}/L = 1.46 \times 10^{-3}) \) for \( \Gamma = 1.00 \), and \( Nu = 642 \) and thus \( \lambda_{th} = 0.30 \text{ mm} \) \( (\lambda_{th}/L = 7.8 \times 10^{-4}) \) for \( \Gamma = 0.50 \).

In figure 6(b,d) we have indicated the location of \( z/\lambda_{th} = 200 \) by vertical dotted...
Logarithmic temperature profiles for a Prandtl number of 12.3

**Figure 6.** (Colour online) Plots of $\Theta(z)$ as a function of $z/L$. (a,b) Results for $\Gamma = 1.00$; the radial position is $R - r = 6.0$ mm (i.e. $\xi = (R - r)/R = 0.062$), the Rayleigh number is $2.07 \times 10^{11}$ and the Nusselt number is 343 (run 1310071, $\Delta T = 24.8$ K). Columns $V_2$ (open circles) and $V_6$ (open squares) (see table 2) and their averages (solid circles) are given. In (b) the data are shown on a logarithmic horizontal scale, and the averages are given also as a function of $1 - z/L$ (solid blue diamonds). (c,d) Results for $\Gamma = 0.50$ (column $V_1$); here the radial position is $\xi = (R - r)/R = 0.063$, the Rayleigh number is $1.33 \times 10^{12}$ and the Nusselt number is 642 (run 1404151, $\Delta T = 20.0$ K). Vertical dashed lines indicate the sample centre at $z/L = 0.5$; horizontal dashed lines represent the non-OB shift of the centre temperature, $\phi = 0.0137$ for $\Gamma = 1.00$ and $\phi = 0.0127$ for $\Gamma = 0.50$. In (b) and (d) the upper horizontal axis is scaled by the thermal boundary-layer thickness, $\lambda_{th} = 1.46 \times 10^{-3}L$ for (b) and $\lambda_{th} = 7.8 \times 10^{-4}L$ for (d), and the vertical dotted lines correspond to $z/\lambda_{th} = 200$. 

$\xi = (R - r)/R$.
Figure 7. (Colour online) Deviations of the data shown in figure 6 from a fit of (1.1) to the points near the bottom plate (solid symbols) or of (1.2) to the points near the top plate (open symbols), as described in the caption of figure 6. In (a) the data are shown as a function of $z/L$ or $1-z/L$, and in (b) they are displayed as a function of $z/\lambda_{th}$ or $(L-z)/\lambda_{th}$, where $\lambda_{th} = L/(2\nu)$ is the thickness of the thermal BL. Red circles represent results for $\Gamma = 1.00$ and blue diamonds results for $\Gamma = 0.50$.

lines. One sees that this is a reasonable estimate of the approximate location of the outer limit of the log layer for both aspect ratios. This becomes more apparent in figure 7(b), where the deviations $\Theta - \Theta_{fit}$ are plotted as a function of $z/\lambda_{th}$ or $(L-z)/\lambda_{th}$. There the vertical dotted line is drawn at 200, and all data to the left of it show only random deviations from zero.

We note that for $\Gamma = 0.50$, an outer layer is found for $z/\lambda_{th} \gtrsim 200$ ($z/L \gtrsim 0.15$); but for $\Gamma = 1.00$, where $z/\lambda_{th} = 200$ corresponds roughly to $z/L = 0.3$, the sample is too short for a significant outer layer to develop. Thus, in most of the shorter sample near-wall turbulence is expected to prevail, and free turbulence is unlikely to be found in this system.

4.3.2. Dependence of the amplitude of the logarithmic term on the Rayleigh number

In figure 8(a,b) the absolute value of the amplitude $A$ of the logarithmic profile is plotted as a function of $Ra$ for $\Gamma = 1.00$. Figure 8(a) is for $\xi = 0.020$ where, as seen in figure 5(a), $|A|$ was exceptionally large and the point nearest the bottom plate at $z/L = 0.020$ was anomalously high. That point was excluded from all fits. Fitting (1.1) to the remaining points at $z/L = 0.043$, 0.100 and 0.210 gave the open squares in (a). They show a significant dependence of $A$ on $Ra$. However, close inspection of the data at smaller $Ra$ (see the asterisks in figure 5a) revealed significant deviations from the fit even for $z/L = 0.043$ (which had been included in the fit). Fitting to only the two points at $z/L = 0.100$ and 0.210 gave the solid squares in the figure. A fit of

$$A(Ra, \xi) = A_0(\xi)Ra^\eta$$

(4.2)

to these data yielded $\eta = -0.011 \pm 0.025$, consistent with a log amplitude that is independent of $Ra$ ($\eta = 0$). We conclude that the $Ra$ dependence suggested by the open squares in the figure is illusory and due to the encroachment of the BL upon the log layer as $Ra$ decreases.

The results at the other radial positions for $\Gamma = 1.00$ are less ambiguous. Fitting (1.1) in each case to the three points at $z/L = 0.020$, 0.043 and 0.100 gave the results in figure 8(b). The data sets at all $\xi$ are consistent with a $Ra$ independent amplitude...
Logarithmic temperature profiles for a Prandtl number of 12.3

Figure 8. (Colour online) The absolute value $|A|$ of the amplitude of the logarithmic term as a function of $Ra$: (a) data for $\Gamma = 1.00$ and $\xi = 0.020$; (b) data for $\Gamma = 1.00$ and, from bottom to top, $\xi = 1.00$, 0.630, 0.210 and 0.063; (c) data for $\Gamma = 0.50$ and, from bottom to top, $\xi = 1.00$, 0.325 and 0.063. The lines are fits of the power law $A(Ra, \xi) = A_0(\xi)Ra^n$ to the data.

of the log profile. This is made more concrete by figure 9, which for $\Gamma = 1.00$ shows all values of $\eta$ as solid circles, together with the probable errors derived from the fits, as a function of the radial position $\xi$. On the basis of this analysis we conclude that for $\Gamma = 1.00$ and $Pr = 12.3$, any $Ra$ dependence of the amplitude of the logarithmic profile is absent or too small to be resolved by the measurements. This differs from the analogous result for $\Gamma = 0.50$ and $Pr \simeq 0.8$ obtained by Ahlers et al. (2014), which yielded $\eta \simeq -0.12$. A $Ra$ independent amplitude of the log profile differs from the prediction based on a multilayer model by She et al. (2014), who obtained $\eta \simeq -0.16$.

Figure 8(c) shows $|A|$ as a function of $Ra$ for $\Gamma = 0.50$ at three values of $\xi$. As was the case for $\Gamma = 1.00$, one also sees here that the measurements do not reveal any $Ra$ dependence of $A$. Fits of (4.2) to the data yielded the solid blue squares in figure 9. The data are consistent with a $Ra$ independent amplitude of the log profile also for $\Gamma = 0.50$.

Since we found no $Ra$ dependence of $A$, we show the averages of the measurements at all $Ra$ for a given $\xi$ in figure 10 as red circles for $\Gamma = 1.00$ and as blue squares for $\Gamma = 0.50$. The data do not reveal any significant $\Gamma$ dependence. The $\Gamma$ independence of $A$ is consistent with the idea that the logarithmic profile depends only on the distance from the plates in an appropriately chosen wall unit (which, as mentioned above, has been suggested by Ahlers et al. 2014 to be the thermal boundary-layer thickness $\lambda_{th}$) and not on the overall length of the sample.
Figure 9. (Colour online) Values of $\eta$ as a function of the radial position $\xi$. Solid black circles represent the exponent $\eta(\xi)$ derived from a fit of $A(Ra, \xi) = A_0(\xi)Ra^\eta$, separately at each $\xi$ value, to the data for $\Gamma = 1.00$ shown as solid symbols in figure 8(a,b). The open circle is $\eta$ derived from a similar fit to the open squares in figure 8(a). The solid blue squares show $\eta$ derived from the data in figure 8(c) for $\Gamma = 0.50$.

Figure 10. (Colour online) The absolute value $|A|$ of the coefficient of the logarithmic term, averaged over $Ra$ separately at each $\xi = (R - r)/R$, on a logarithmic scale as a function of $\xi$ on a logarithmic scale. The solid line is the function $A(\xi) = A_1/\sqrt{2\xi - \xi^2}$ (Grossmann & Lohse 2012) with $A_1 = 0.00155$. The dotted, dashed and dash–dot lines are the DNS results from Ahlers et al. (2012) for $Ra = 2 \times 10^{12}$, $Ra = 2 \times 10^{11}$ and $Ra = 2 \times 10^{10}$, respectively, all for $\Gamma = \frac{1}{2}$ and $Pr = 0.7$. The solid diamonds are results from Ahlers et al. (2014) for $\Gamma = 0.50$, $Pr = 0.786$ and $Ra = 2 \times 10^{12}$.

The solid line in figure 10 corresponds to the function

$$A(\xi) = A_1/\sqrt{2\xi - \xi^2},$$

(4.3)

which was proposed by Grossmann & Lohse (2012) for the ultimate state but which also serves as a convenient representation of our results in the classical state when we choose $A_1 = 0.00155$. Obviously the point at $\xi = 0.020$, derived from the solid squares in figure 8(a), deviates from this representation, presumably because of the proximity of the viscous boundary layer along the sidewall, as discussed above.
Also shown in figure 10 for comparison are the DNS results for $Pr = 0.7$, which, from bottom to top, are for $Ra = 2 \times 10^{12}, 2 \times 10^{11}$ and $2 \times 10^{10}$ (dotted, dashed and dash–dot lines, respectively) (Stevens, Lohse & Verzicco 2011, as reported by Ahlers et al. 2012). In contradistinction to our results for $Pr = 12.3$, the DNS results depend on $Ra$. For $Ra = 2 \times 10^{12}$, they agree well with the measurements by Ahlers et al. (2014) at that $Ra$ and $Pr \approx 0.8$ (the solid diamonds in the figure). The experimental data for $Pr \approx 0.8$ also revealed a $Ra$ dependence (Ahlers et al. 2014), with $A \approx Ra^{-0.12}$. Thus, it seems that the $Ra$ independent amplitude of the logarithmic profile prevails only at large $Pr$.

Based on a model where the plumes are responsible for creating the logarithmic profile, one would expect that the radial dependence of $A$ would not prevail in samples of sufficiently large $\Gamma$. For large $\Gamma$, there are many large-scale circulation (LSC) cells distributed over the horizontal plane (see, for instance, Bailon-Cuba, Emran & Schumacher 2010), and the smaller-scale turbulent fluctuations drive this complex LSC structure stochastically (Brown & Ahlers 2007b, 2008), leading to random lateral diffusion of the upflow and downflow boundaries (Hogg & Ahlers 2013). Assuming that the plumes continue to be concentrated near these boundaries, the time-averaged plume density (and thus the log amplitude $A$) will be more nearly uniform throughout the sample.

4.4. The stabilizing gradient at the sample centre

The data in figure 5 for $\xi = 0.062$ and 1.00 at $z/L = 0.500$, as well as the measurements with the eight thermistors located in the sidewall in the horizontal midplane (see § 4.1), show that there is no measurable horizontal temperature variation in the sample at half-height ($z/L = 0.500$). However, it is interesting to note that for $\Gamma = 1.00$, the temperatures at $z/L = 0.210$ for $\xi = 0.630$ and 1.00 are actually lower than the temperature at the sample centre, revealing a stabilizing temperature gradient in that location. This is seen more clearly in figure 11(a), where the data near the vertical sample axis are shown on a linear scale. Stabilizing temperature gradients in the bulk of RBC for $\Gamma = 1.00$ have been measured before by Tilgner et al. (1993) for $Pr = 6.6$ and by Brown & Ahlers (2007b) for $Pr = 4.4$ and 5.5. They have also been observed in DNS (Schmalzl, Breuer & Hansen 2002; Breuer et al. 2004), albeit for samples with different $\Gamma$ and square cross-sections. For comparison with the measurements of Brown & Ahlers (2007b), we estimate that $\Delta \Theta_0/\Delta (L/z) = [\Theta(z_1/L) - \Theta(z_0/L)]/(z_1/L - z_0/L)$ with $z_1/L = 0.500$ and $z_0/L = 0.210$. This quantity is comparable to $-\Delta T_0/\Delta T$ of Brown & Ahlers (2007b). The results for $\xi = 1.00$ (respectively, 0.63) are shown as solid circles (respectively, squares) as a function of $Ra$ in figure 12, together with the data of Tilgner et al. (1993) and Brown & Ahlers (2007b). Although the results are not very accurate because the effect is so small, one sees that the trend towards a smaller negative gradient with increasing $Ra$ continues at the larger $Ra$. The line in the figure is drawn with a slope of $-0.5$, suggesting that $\Delta \Theta_0/\Delta (z/L) \propto Ra^{-0.5}$; but obviously the uncertainty of the exponent is quite large, say near $\pm 0.1$ or so. Of course there may also be some unresolved $Pr$ dependence which, if taken into account, could alter the $Ra$ dependence deduced from the data.

The data in figure 11(b) suggest that there is no stabilizing temperature gradient for our $\Gamma = 0.50$ sample.
5. Summary

The main purpose of this paper is to present new measurements of the amplitude $A$ of the logarithmic ‘near-wall’ temperature profile (the ‘log’ profile) in the bulk of turbulent RBC for a Prandtl number of $Pr = 12.3$ and for the two aspect ratios $\Gamma = 0.50$ and 1.00. The measurements covered the range $2 \times 10^{10} \leq Ra \leq 2 \times 10^{11}$ for $\Gamma = 1.00$ and the range $3 \times 10^{11} \leq Ra \leq 2 \times 10^{12}$ for $\Gamma = 0.50$. While the log profile in RBC was first discovered for $Pr \simeq 0.8$ (Ahlers et al. 2012, 2014) and $\Gamma = 0.5$, there had been no prior information about its $Pr$ and $\Gamma$ dependence (if any).

Although non-OB effects led to a small shift in the centre temperature $T_c$ of the sample away from its mean temperature $T_m$, we found no significant difference within our resolution between the log amplitudes $A$ and $A'$ near the bottom and the top plates. A similar result was found for $Pr = 0.8$ in the classical state, but not in the ultimate state where the measurements yielded $A/A' \simeq 0.95 \pm 0.01$ (Ahlers et al. 2014). The result $A = -A'$ is consistent with the view that the log profiles are a property of the bulk, and that (in the classical state) non-OB effects lead to different temperature drops across the top and bottom thermal BLs but do not influence the properties of the bulk very much (see also Brown & Ahlers 2007b).
Within our resolution, we found no $\Gamma$ dependence of $A$. Based on a model where the log profile is generated by plumes emitted from the top and bottom BLs (see § 4.3 of Ahlers et al. 2014), this is a reasonable result. The profile would be expected to depend only on the distance from the plates, regardless of the overall length of the sample.

As had been found also for $Pr \simeq 0.8$, $A$ was strongly dependent upon the radial distance from the sidewall. Near (but not too near) the wall, this dependence could be represented by the power law $A \simeq A_0 \xi^{-0.5}$ where $\xi = 1 - r/R$. For $Pr \simeq 0.8$, $A$ was larger by a factor of 2 or so than our result for $Pr=12.3$, and decreased slightly more rapidly with increasing $\xi$. The dependence of $A$ on $\xi$ is qualitatively consistent with the surmise that the log profile is generated by the plumes emitted from the thermal BLs. For a cylindrical sample of aspect ratio near 1, the plume density is well known to be largest near the sidewall and smallest along the vertical centreline. One would expect the radial dependence of $A(\xi)$ to diminish as $\Gamma$ becomes large, because for large $\Gamma$ the sample will contain many LSC cells which diffuse randomly in the horizontal directions, thus providing a horizontal averaging of the plume density and, one assumes, also of the log profiles.

An interesting difference between the two Prandtl numbers is the dependence of $A$ and $A'$ upon $Ra$. For the larger Prandtl number $Pr = 12.3$, we found no $Ra$ dependence of $A$ for either aspect ratio. This differs from the finding for $Pr \simeq 0.8$, where $A \propto Ra^{-\eta}$ with $\eta \simeq 0.12$. Ahlers et al. (2014, § 4.3.1) considered a two-sublayer mean-field model of the temperature field in turbulent RBC, which gave $A \propto \left(\lambda_u/\lambda_{th}\right) Re_s^{-\frac{1}{2}}$ (here $\lambda_u$ is the viscous boundary-layer thickness and $Re_s$ is the shear Reynolds number of the viscous BL). Measurements of and DNS for $\lambda_u/\lambda_{th}$ (du Puits, Resagk & Thess 2009; Stevens et al. 2011), as well as analytical arguments for Prandtl–Blasius BLs (Shishkina et al. 2010), gave a $Ra$ independent $\lambda_u/\lambda_{th}$. For $Re_s$ one expects $Re_s \sim Re_s^4$ (Grossmann & Lohse 2001), with $Re \propto Ra^{0.423}$ (He et al. 2012), yielding $A \sim Ra^{-\eta_{th}}$ with $\eta_{th} \simeq 0.105$. This agrees well with the experiment for $Pr = 0.8$, but does not agree with our $Ra$ independent $A$ for $Pr = 12.3$. However, the Grossmann–Lohse model (Stevens et al. 2013) gives $\lambda_u/\lambda_{th} \propto Ra^{0.10}$ nearly independent of $Pr$, which by the above arguments leads to a nearly $Ra$ independent $A$ as found in the present work but in disagreement with the result for $Pr = 0.8$. Thus we unfortunately have to conclude that the theoretical results for the $Ra$ dependence of $A$ remain somewhat ambiguous.

Very near the wall, the amplitude of the log profile was larger than expected from an extrapolation of the data further away. We attribute this to the closeness of the viscous sidewall BL, but do not know why this proximity should lead to an increase of $A$.

In addition to the results for the log profile, we also reported measurements of the Nusselt number and of the non-OB effects on the centre temperature $T_c$. As expected, we found that $T_c - T_m$ at constant applied temperature difference (where $Ra$ for the two $\Gamma$ values differed by a factor of 8) was independent of $\Gamma$ within the experimental resolution. The observed non-OB effects were small enough to have a negligible influence on the Nusselt measurements.

The results for $Nu$ were used to estimate the thermal boundary-layer thickness $\lambda_{th} \equiv L/(2Nu)$, which, we believe, is the natural inner length scale for the log profiles in RBC. We found that $Nu$ for the two aspect ratios at the same $Ra$ differed by less than 1%, a result consistent with previous comparisons of $Nu$ for different $\Gamma$. The measurements were consistent with the model of Grossmann and Lohse, which gives $Nu(Ra, Pr)$ for $\Gamma = 1$. 

Logarithmic temperature profiles for a Prandtl number of 12.3
Finally, we examined the time-averaged vertical temperature gradient near the centre of the sample. For $\Gamma = 1.00$, we found this gradient to be stabilizing, as had been observed before for $\Gamma = 1.00$ and $Pr$ in the range from 4.4 to 6.6. The magnitude of the gradient decreased with increasing $Ra$. For $\Gamma = 0.50$, we found no evidence of a stabilizing gradient.

Acknowledgement
This work was supported by the US National Science Foundation through grant DMR11-58514.

Appendix. Locations of the thermistors in the bulk of the samples

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<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>V₁</td>
<td>6.0</td>
<td>0.063</td>
<td>1</td>
</tr>
<tr>
<td>V₂</td>
<td>11.0</td>
<td>0.116</td>
<td>2</td>
</tr>
<tr>
<td>V₃</td>
<td>31.0</td>
<td>0.325</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1. The radial and azimuthal locations of the thermistor columns for the samples with $\Gamma = 1.00$ (left columns) and 0.50 (right columns). The angle $\Phi$ is the azimuthal location of the thermistor column relative to an arbitrary origin; the sample height is $L = 190$ mm for $\Gamma = 1.00$ and $L = 380$ mm for $\Gamma = 0.50$, and the radius is $R = 95$ mm for both samples.

<table>
<thead>
<tr>
<th>z/L</th>
<th>V₀</th>
<th>V₁</th>
<th>V₂</th>
<th>V₃</th>
<th>V₄</th>
<th>V₅</th>
<th>V₆</th>
<th>V₇</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.020</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0.043</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0.100</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0.210</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0.500</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0.790</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0.900</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0.960</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

Table 2. The vertical locations of the thermistors in the eight columns defined in table 1. An ‘X’ in a given column indicates that that column contained a thermistor at that vertical position.
REFERENCES


BOUSSINESQ, J. 1903 *Theorie Analytique de la Chaleur*, vol. 2. Gauthier-Villars.


