Heat-transport enhancement in rotating turbulent Rayleigh-Bénard convection

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We present new Nusselt-number (Nu) measurements for slowly rotating turbulent thermal convection in cylindrical samples with aspect ratio \( \Gamma = 1.00 \) and provide a comprehensive correlation of all available data for that \( \Gamma \). In the experiment compressed gasses (nitrogen and sulfur hexafluoride) as well as the fluorocarbon \( \text{CF}2\text{Cl2} \) (3M Fluorinert FC72) and isopropanol were used as the convecting fluids. The data span the Prandtl-number (Pr) range 0.74 < Pr < 35.5 and are for Rayleigh numbers (Ra) from 3 \( \times \) 10^6 to 4 \( \times \) 10^9. The relative heat transport \( \text{Nu}, (1/\text{Ro}) \equiv \text{Nu}(1/\text{Ro})/\text{Nu}(0) \) as a function of the dimensionless inverse Rossby number 1/Ro at constant Ra is reported. For \( \text{Pr} \approx 0.74 \) and the smallest \( \text{Ra} = 3.6 \times 10^6 \) the maximum enhancement \( \text{Nu}_{\text{max}} - 1 \) due to rotation is about 0.02. With increasing \( \text{Ra} \), \( \text{Nu}_{\text{max}} \) - 1 decreased further, and for \( \text{Ra} \geq 2 \times 10^7 \) heat-transport enhancement was no longer observed. For larger \( \text{Pr} \) the dependence of \( \text{Nu} \), on 1/Ro is qualitatively similar for all \( \text{Pr} \). As noted before, there is a very small increase of \( \text{Nu} \), for small 1/Ro, followed by a decrease by a percent or so, before, at a critical value 1/Ro, a sharp transition to enhancement by Ekman pumping takes place. While the data revealed no dependence of 1/Ro on \( \text{Ra} \), 1/Ro, decreased with increasing \( \text{Pr} \). This dependence could be described by a power law with an exponent \( \alpha \approx 0.41 \). Power-law dependencies on \( \text{Pr} \) and \( \text{Ra} \) could be used to describe the slope \( S_\text{Nu} = \partial \text{Nu}/\partial (1/\text{Ro}) \) just above 1/Ro. The \( \text{Pr} \) and \( \text{Ra} \) exponents were \( \beta_1 = -0.16 \pm 0.08 \) and \( \beta_2 = -0.04 \pm 0.06 \), respectively. Further increase of 1/Ro led to further increase of \( \text{Nu} \) until it reached a maximum value \( \text{Nu}_{\text{max}} \). Beyond the maximum, the Taylor-Proudman (TP) effect, which is expected to lead to reduced vertical fluid transport in the bulk region, lowered \( \text{Nu} \), \( \text{Nu}_{\text{max}} \), was largest for the largest \( \text{Pr} \). For \( \text{Pr} = 28.9 \), for example, we measured an increase of the heat transport by up to 40% (\( \text{Nu} - 1 = 0.40 \)) for the smallest \( \text{Ra} = 2.2 \times 10^6 \), even though we were unable to reach \( \text{Nu}_{\text{max}} \) over the accessible 1/Ro range. Both \( \text{Nu}_{\text{max}}(\text{Pr}, \text{Ra}) \) and its location 1/Ro, along the 1/Ro axis increased with \( \text{Pr} \) and decreased with \( \text{Ra} \). Although both could be given by power-law representations, the uncertainties of the exponents are relatively large.

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I. INTRODUCTION

In addition to being of major fundamental interest in fluid mechanics, thermal convection in a rotating reference frame has been a topic of research for many years because it occurs in many geo- and astrophysical systems. Important examples occurring in nature include atmospheric flows in solar planets [see, for instance, Ref. [1]], convection in an outer portion of the sun [2] which determines the temperature on Earth, and the flow of conducting material in Earth’s outer core which generates the magnetic field that protects us from cosmic radiation [3].

Studies of thermal convection in the laboratory or in numerical simulations use or consider a fluid which is confined by a cold horizontal plate from above and a parallel warm plate from below. This system is known as Rayleigh-Bénard convection or RBC (for a general introduction to RBC, see, for instance, [4,5]; for more detailed reviews see, e.g., [6–8] and references therein). The vertical temperature difference across the sample is expressed in dimensionless form by the Rayleigh number Ra [see Eq. (3) below]. For sufficiently large Ra the fluid flow is turbulent [9–13] and the heat transport, expressed in dimensionless form as the Nusselt number Nu [see Eq. (8) below], is strongly enhanced by the fluid motion. In this case, most of the temperature drop across the sample takes place within two thin thermal boundary layers (BLs), one below the top and the other above the bottom plate [14–17], while the temperature in the bulk of the sample, even though vigorously fluctuating, remains nearly constant in the time average (see, however, Refs. [18–22] for more detailed descriptions). In the bulk there are large-scale flow structures (the “large-scale circulation” or LSC) in addition to the fluctuations on a wide range of smaller scales [23]. In the case of a sample with height \( L \) close to its diameter \( D \) the LSC takes the form of a single convection roll with up flow and down flow along the side wall on opposite sides. The LSC has a dynamic characteristic of a stochastically driven system [24,25]. The circulation plane of the LSC diffuses azimuthally, driven by the small-scale fluctuations. The LSC amplitude varies irregularly in time, and occasionally undergoes a cessation where it briefly vanishes completely. All the work discussed in the present paper is about turbulent convection in a cylindrical sample with aspect ratio \( \Gamma \equiv D/L = 1.00 \).

Rotating a sample of turbulent RBC about its vertical axis at an angular frequency \( \Omega \) introduces a host of interesting new phenomena (for a recent review, see [26]). In dimensionless form \( \Omega \) is usually expressed by the inverse Rossby number 1/Ro [see Eq. (6)] or the inverse Ekman number 1/Ek [see Eq. (7)], both of which are proportional to \( \Omega \). Constant rotation affects the flow field, and thus Nu, because it couples to the velocity and thereby introduces the Coriolis force. It also introduces a centripetal force, but this is often neglected in a theoretical analysis and experiments tend to be designed so as to minimize its influence.

Rotation stabilizes the pure conduction state and thus leads to an increase of the critical Rayleigh number \( Ra_c(\Omega) \) for the onset of convection [27–32]. For somewhat larger Ra this
stabilizing influence is reflected in a reduction of the heat flux when the rotation rate is increased at constant $Ra > Ra_c$. However, when $Ra$ is large enough for the convection to be turbulent, slow rotation (small $1/Ro$ or $1/Ek$) at constant $Ra$ leads to an enhanced $Nu$ [29,30,33]. We shall describe this phenomenon in terms of the relative Nusselt number,

$$Nu_r = Nu(1/Ro)/Nu(0).$$

(1)

It is believed that this enhancement is due to Ekman pumping that occurs when thermal plumes that emerge from the thermal boundary layers are turned into vortices by the Coriolis force; these vortices pump hot or cold fluid from the bottom or top boundary layers into the bulk regions [29].

A remarkable recent experimental discovery was that the heat-flux enhancement does not occur for arbitrarily small rotation rates. Rather, $1/Ro$ has to exceed a critical value $1/Ro_c$ before Ekman pumping sets in [34]. At $1/Ro_c$ there is a transition between two different turbulent states, one without and the other with Ekman vortices. The value for $1/Ro_c$ depends on $\Gamma$ and the experimental data suggest that it vanishes as $\Gamma$ tends toward infinity, thus leaving the unbounded state without this transition [34,35]. A Ginzburg-Landau-like model was developed that explains the transition as a result of the influence of the lateral boundaries on the vortex formation inside the fluid [36,37]. In this model, the transition is a forward bifurcation, causing a discontinuity of the slope,

$$S_{Ro} = (\partial Nu_r/\partial (1/Ro))_{Ra},$$

(2)

at $1/Ro_c$.

Above $1/Ro_c$ the heat transport first increases linearly with $1/Ro$, but as $1/Ro$ becomes larger $S_{Ro}$ decreases and eventually becomes negative, thus leading to a maximum of $Nu_r(1/Ro)$ at constant $Ra$ and $Pr$. Qualitatively this phenomenon is due to a competition between Ekman pumping (which enhances $Nu$) and the stabilizing influence due to the Taylor-Proudman (TP) effect (see, e.g., Ref. [38]) which tends to suppress vertical velocity gradients and thereby introduces temperature gradients in the bulk and reduces $Nu$. The state of the system in the parameter range where $Nu_r > 1$ because Ekman pumping is more effective than TP suppression often is referred to as buoyancy-driven turbulence. The current investigation focuses on this parameter range.

It had been thought that above $1/Ro_c$, $Nu_r(1/Ro)$ at constant $Ra$ evolves smoothly with increasing $1/Ro$ in the buoyancy-driven regime without any further transitions. However, recently it was discovered [39] that there actually is a sequence of transitions where $S_{Ro}$ changes discontinuously while, within the resolution of the experiment, $Nu_r$ remains continuous. Between these transitions $Nu_r$ is well approximated by a linear dependence on $1/Ro$. For modest $Ra$ the slope changes at the transitions are not very large and were not easily recognized in the data; but at $Ra = O(10^4)$ and larger the transitions became quite obvious. For the analysis of the present paper we shall not take the transitions beyond $1/Ro_c$ into consideration and focus only on the dependence on $Ra$ and $Pr$ of $1/Ro_c$, of the slopes $S_{Ro}$ and $S_{Ro_c}$ immediately below and above $1/Ro_c$, and of the height $Nu_{r,max}$ and location $1/Ro_{max}$ along the $1/Ro$ axis of the maximum of $Nu_r$.

With further increase of $1/Ro$ the system enters a rotation-dominated regime where the TP suppression of the heat transport is larger than the Ekman-pumping enhancement (i.e., $Nu_r < 1$) [see, e.g., Refs. [29–31,33,40–42]]. When $1/Ro$ becomes even larger (but $Ra$ is large enough for the system to remain in a turbulent state), one finds the so-called geostrophic flow regime where $Nu_r$ is diminished to values well below one and where the Coriolis force is balanced by pressure gradients [see, e.g., Refs. [41–44]]. While this regime is of exceptional interest in geophysics, astrophysics, atmospheric science, and oceanography, it is difficult to reach experimentally and beyond the scope of the present paper.

There is no uncertainty about the existence of the above-mentioned regimes; but many details are still unclear, as, for example, the influence of $Pr$ on the heat-transport enhancement at small $1/Ro$. Here we consider large $Ra$ and small rotation rates and investigate the heat-transport enhancement in this buoyancy-driven regime. A great deal of work already had been done on this problem using experiments (e.g., Refs. [37,45–50]) and direct numerical simulations (DNS) (e.g., Refs. [34,51,52]); our measurements add to the understanding of the $Pr$ and $Ra$ dependencies of the observed phenomena. Our analysis of all the available data for $\Gamma = 1.00$ provides a unified interpretation.

We found that $1/Ro_c$ is, within the uncertainty of the data, independent of $Ra$ and increases with increasing $Pr$. This $Pr$ dependence can be described by the power law $1/Ro_c \propto Pr^\alpha$ with $\alpha \approx -0.41$. The slope $S_{Ro}$ of $Nu_r$ just below $1/Ro_c$ is negative and has a tendency to become more so with increasing $Ra$, but the data are not good enough to warrant a description in terms of a power law. The initial slope $S_{Ro}$ immediately above $1/Ro_c$ is positive and decreases slightly with increasing $Pr$ and $Ra$. The data can be represented well by a power law of the form $S_{Ro} \propto Pr^{-0.16} \times Ra^{-0.04}$. The maximum heat-transport enhancement $Nu_{r,max}$ decreases with increasing $Ra$ at constant $Pr$, and the decrease can be described by the power law $(Nu_{r,max} - 1) \propto Ra^{\delta_2}$ with $\delta_2 = -0.35$. With increasing $Pr$ at constant $Ra$ $Nu_{r,max}$ increases. Although the data are not so definitive, we assumed also for the $Pr$ dependence a power law $(Nu_{r,max} - 1) \propto Pr^{\delta_1}$, but the fitted exponent value $\delta_1 \approx 0.65$ is subject to a large uncertainty. For $Pr = 28.9$ and $Ra = 2.2 \times 10^9$ we measured an increase of the heat transport by as much as 40%, even though our accessible range of $1/Ro$ was not sufficient to actually reach the maximum. In contrast, for $Pr = 0.74$ the heat transport increased by not more than 2% for the smallest $Ra$, and no enhancement could be measured for larger $Ra$.

In the next section we define the relevant dimensionless parameters needed to describe this system. After that, in Sec. III, we describe the general features of the heat-transport enhancement by using previous measurements taken with water ($Pr = 4.38$) and published in Ref. [47]. We then briefly describe the experimental apparatus in Sec. IV. Section V presents measurement results and data analysis. The paper ends with a summary and discussion where we present our present understanding of the observed phenomena in the weakly rotating regime.

II. RELEVANT PARAMETERS

Investigating thermal convection is usually done in cylindrical vessels of diameter $D$ and height $L$ that are terminated
by a warm plate of temperature $T_b$ from the bottom and a colder plate at temperature $T_f$ from above. If the temperature difference $\Delta T = T_b - T_f$ is sufficiently small, fluid properties can be assumed to be constant throughout the cell and the system is described by the Boussinesq equations [53–55]. In these equations, two dimensionless control parameters occur. They are the Rayleigh number,

$$Ra = \frac{\alpha g \Delta T L^3}{\nu \kappa},$$

and the Prandtl number,

$$Pr = \frac{\nu}{\kappa}.$$

Here $\alpha$, $\kappa$, and $\nu$ denote the isobaric thermal expansion coefficient, the thermal diffusivity, and the kinematic viscosity of the fluid. These properties are evaluated at the mean temperature $T_m = (T_b + T_f)/2$. The gravitational acceleration is $g$. For a cylindrical sample a third relevant parameter is the aspect ratio,

$$\Gamma \equiv D/L.$$

The value of $Ra$ for which the flow becomes turbulent depends on $Pr$. An increase of $Pr$ stabilizes the flow, i.e., the transition to turbulence is shifted to larger values of $Ra$ [9–11,13]. All measurements considered in this paper, including those at the larger $Pr$ values, are done in the fully turbulent regime.

From the definitions of $Ra$ and $Pr$ we see that the first parameter includes the buoyancy and the damping forces), while the second indicates whether thermal diffusion or viscosity is the main reason for the damping of the flow.

When the sample rotates around its vertical axis, the angular rotation frequency $\Omega$ (in rad/s) is an additional control parameter that can be varied. There are several dimensionless parameters that can be used in the governing equations to express the effect of $\Omega$. Here we shall use the Rossby number,

$$Ro = \frac{\sqrt{ag \Delta T L}}{2\Omega},$$

which describes the ratio between the buoyant and Coriolis forces and the Ekman number,

$$Ek = \frac{\nu}{\Omega L^2} = 2Ro \sqrt{\frac{Pr}{Ra}},$$

which compares the viscous with the Coriolis force. While either of these two numbers is sufficient to describe the system, one or the other may be more appropriate depending on parameter values and issues of interest. Following the convention of recent previous publications, we shall use the inverse Rossby number $1/Ro$ and the inverse Ekman number $1/Ek$ because these are proportional to $\Omega$.

An important global property usually investigated is the heat flux from the bottom to the top plate as expressed by the dimensionless Nusselt number,

$$Nu = \frac{Q L}{\lambda A \Delta T}.$$

This quantity relates the overall heat-current density $Q/A$ to the purely conductive heat flux $\lambda \Delta T/L$ that would occur without convection. Here, $\lambda$ is the thermal conductivity of the fluid.

In Eq. (2) we already defined the slope $S_{Ro}$ of $Nu (1/Ro)$. Below we shall also use the slope,

$$S_{Ek} = [\partial Nu/\partial (1/Ek)]_{Ro}.$$  

From Eq. (7) one sees that

$$S_{Ek} = 2S_{Ro} \sqrt{Pr/Ra}.$$

III. GENERAL FEATURES OF THE HEAT TRANSPORT AS A FUNCTION OF $1/Ro$

Before proceeding to the new results of the present investigation, we provide an overview of the different ranges of heat-transport enhancement in rotating buoyancy-driven turbulent convection and review the extent to which the various encountered phenomena are understood.

Figure 1 shows typical data sets [47] for $Nu$, as a function of $1/Ro$ for $Pr = 4.38$ at constant $Ra$. Figures 1(a) and 1(b) show data for a relatively small $Ra = 2.2 \times 10^9$ and Figs. 1(c) and 1(d) display measurements for the larger $Ra = 1.8 \times 10^{10}$. For Figs. 1(a) and 1(c) a linear horizontal scale was used, while Figs. 1(b) and 1(d) are displayed on logarithmic horizontal scales in order to show more clearly the dependencies at small $1/Ro$. One sees that $Nu$ is not a monotonic function of $1/Ro$, and that the slope $S_{Ro}$ [see Eq. (2)] changes with $(1/Ro)$. For both $Ra$, one can identify three ranges, separated by vertical dashed lines and marked by roman numbers in Figs. 1(b)–1(d).

At small $Ra$ and $1/Ro$, $Nu$ remains nearly constant and very close to one with increasing $1/Ro$ [range I in Fig. 1(b)]. In range II $Nu$ increases as $1/Ro$ increases. From a Ginzburg-Landau-like model, we know that the transition between ranges I and II at $1/Ro$ is sharp (see Sec. I and [36,37]) and the increase of $Nu$ beyond it is initially (locally) linear. At larger $1/Ro$ the slope decreases and $Nu$ reaches a maximum $Nu_{max}$ at $1/Ro_{max}$. Beyond $1/Ro_{max}$ $Nu$ decreases again (range III).

When with further increase of $1/Ro$ $Nu$ has decreased to the point where it is less than one, it eventually enters the geostrophic regime and thereafter the pure conduction state where $Nu = 1/Nu(0) \ll 1$ and $Nu = 1$; these ranges are not under consideration in the current paper.

While the three ranges indicated in Fig. 1(b) are clearly distinguishable at any $Ra$, for larger $Ra$ the structure becomes more complex and we divide the ranges into subranges. As can be seen in Figs. 1(c) and 1(d), for larger $Ra$ $Nu$ is only (nearly) constant for very small $1/Ro$ and then decreases towards the end of range I. Thus we define two subranges Ia and Ib. One sees that there is a reduction of the heat transport due to rotation in range Ib which, to our knowledge, has not yet been elucidated. This range extends up to $1/Ro$, where Ekman pumping starts and $Nu_{max}$ begins to increase.

Remarkably, another seemingly discontinuous change of the slope $S_{Ro}$ is seen in Fig. 1(c) at $1/Ro \approx 0.8$ (marked by an arrow). The same transition becomes more pronounced for larger $Ra$ and could also be observed in the bulk temperature gradient in the fluid [39]. While in ranges II and III $S_{Ro}$ changes smoothly with increasing $1/Ro$ when $Ra = 2.2 \times 10^9$ [Figs. 1(a) and 1(b)], at larger $Ra$ different regions of
FIG. 1. Relative Nusselt numbers $N_u$ as a function of $1/R_o$ (bottom $x$ axis) and $1/E_k$ (top $x$ axis) for $Pr = 4.38$, and $Ra = 2.2 \times 10^9$ [(a) and (b)] and $Ra = 1.8 \times 10^{10}$ [(c) and (d)]. The short-dashed vertical lines in (b), (c), and (d) mark $1/R_o$ values at which the slope $\partial N_u/\partial (1/R_o)$ changes sign, separating different $1/R_o$ intervals. The dashed red and dash-dotted green lines in (c) show fits of linear functions based on data points close to the minimum at $1/R_o$. The black solid lines are fits of Eq. (15) to the data points close to the maximum $1/R_{omax}$. The boundary between ranges I and II corresponds to a sharp transition at $1/R_o$. The arrow in (c) marks a second transition that was recently discovered and discussed in [39]. Plots (b) and (d) show the same data as (a) and (c), but on logarithmic $x$ axes. The uncertainties of $N_u$ in these plots are indicated by the scatter of the data points and are smaller than the symbol sizes. The figure is based on data published in Ref. [47].

essentially constant slopes are separated by further rather sharp transitions [39].

A similar nonmonotonic dependence on $1/R_o$ was found as well for several other quantities, such as the time-averaged LSC temperature amplitude $\delta$, the LSC cessation frequency, the rotation rate of the LSC circulation plane, and the temperature gradient near the side wall of the sample [47].

The reasons for the existence of the different ranges is understood only partially and mostly qualitatively. In range I the system does not differ in a major way from the case of no rotation in the sense that there still exists a large-scale circulation, and thermal boundary layers are still located adjacent to the plates and sustain most of the temperature difference. However, a weak Coriolis force is acting upon this system. Since in this parameter range the heat transport is determined primarily by the thickness of the thermal boundary layers, one might look for an explanation of the substructure of $N_u$ and its $Ra$ dependence in range I by considering the influence of the rotation on the boundary layers. This was done in Ref. [51]; but these authors focused primarily on the influence of a weak centripetal force on the BLs (which is usually neglected in the theory of rotating convection), and their numerical calculations were for the relatively small $Ra = 4 \times 10^7$. While they found a gradual increase of $N_u$ with $1/R_o$, their model has not explained the nonmonotonic structure which was found experimentally at larger $Ra$ and which leads to the two ranges Ia and Ib.

Interesting results of the flow structure were obtained also in experiments with cylinders of aspect ratio $\Gamma = 1/2$ [48]. There, range I is larger and changes of the LSC properties in this range have been observed, such as the azimuthal rotation rate of its circulation plane and the frequency at which the LSC temporarily breaks up into two rolls, one on top of the other. However, due to the differences in the LSC structure for $\Gamma = 1$ and $\Gamma = 1/2$ it is unclear to which extent there is a relationship to the phenomena discussed here for $\Gamma = 1.00$.

Another attempt to understand this range of weak rotation was presented in Ref. [56], which extended the model of Brown and Ahlers for the flow structure of RBC without rotation [24,25,57] to the rotating case. However, while that work provided an interesting relationship between a number of observables, including the frequency of cessations, the amplitude, and the azimuthal velocity fluctuations of the LSC, its predictions were based on model parameters derived from fits to the experimentally determined probability distribution functions of the LSC temperature amplitudes (J.-Q. Zhong and G. Ahlers, unpublished) which already contained the basic nonmonotonic dependence on $1/R_o$. Thus we conclude that even the qualitative features of this range of weak rotation in region I have not been elucidated so far.
Qualitatively, the heat-transport enhancement for $1/Ro > 1/\Gamma^0_{Ro}$ in range II is attributed to the formation of vortices close to the top and bottom plates which form due to the action of the Coriolis force on plumes that are emitted from the BLs \cite{29}. The vortices pump hot (cold) fluid from the bottom (top) boundary layer into the bulk region and thereby enhance the heat transport—a phenomenon referred to as Ekman pumping. The sharp onset for this effect at $1/Ro$ is attributed to the influence of the sidewalls on the formation of these vortices. Experiments and numerical simulations with cylinders of different aspect ratios have shown that $1/Ro_\ast$ increases with decreasing aspect ratio \cite{35–37}. When $\Gamma_\ast$ and/or $1/Ro$ are too small, vortices cannot form and heat transport is not enhanced by Ekman pumping. In analogy to equilibrium critical phenomena this finite-size effect could be described well by a Ginzburg-Landau model \cite{36,37}.

With increasing rotation rate at constant $Ra$, the vortices reach deeper into the fluid and a stronger Coriolis force suppresses the turbulent motion via the TP effect. Therefore, the heat transport decreases with increasing rotation rate in range III. With further increase of the rotation rate the enhanced Taylor-Proudman effect suppresses more vertical fluid motion and thus the heat transport inside the bulk. Instead of the boundary layers, now the bulk becomes the bottleneck for the heat transport and $Nu_\ast < 1$.

Eventually the system enters the geostrophic regime, where pressure gradients are to leading order balanced by Coriolis forces (not shown in Fig. 1). Beyond that, at the highest rotation rates, the conduction state is found with $Nu_\ast \ll 1$ and $Nu = 1$. The range where $Nu_\ast \lesssim 1$ is beyond the scope of the present paper.

IV. EXPERIMENTAL SETUP AND DATA ANALYSIS

We used cylindrical cells with aspect ratios $\Gamma \approx 1$. The experiments were done using two different apparatus; both were described in previous publications. For small-$Ra$ measurements, we used the small convection apparatus (SCA). The design of this apparatus is such that it can be used with two kinds of convection cells, one that can sustain high pressures to be used with compressed gases ($Pr \lesssim 1$, see, e.g., Ref. \cite{58}) and another one for liquids ($Pr > 3$, e.g., Ref. \cite{59}). The apparatus was mounted on a rotating table.

For small-$Pr$ experiments the sample cell was filled with either nitrogen ($N_2$) at 34.5 bars and $T_m = 40$ °C or sulfur hexafluoride ($SF_6$) at 15.2 bars and $T_m = 30$ °C, resulting in $Pr = 0.74$ and $Pr = 0.84$, respectively. The sample cell was $L = 9.84$ cm high and had an inside diameter of $D = 10.16$ cm (resulting in $\Gamma = 1.03$). While the bottom plate was made of copper, the top plate was a 2.54-cm thick single-crystal sapphire disk which allowed optical access (not used in the experiments reported here). We note, that although sapphire has a significantly lower heat conductivity than copper (i.e., the bottom plate), its conductivity is still more than three orders of magnitude larger than that of the working gas. Thus, assuming the top and bottom boundaries at constant temperature is a sufficient approximation. The sidewall was made of high-tensile-strength stainless steel.

The relatively large heat conductivity of the sidewall required a relatively large correction. We corrected the measured heat transport by subtracting the heat transport measured with an evacuated cell. However, other effects of sidewall forcing (see Refs. \cite{60,61}) were neglected. They are of lesser importance in the case where the conductivity of the sidewall is much larger than that of the fluid, which is the case here. Further, since we are interested primarily in the relative change of the heat transport due to rotation, modest systematic errors due to the sidewall conductivity do not play an important role.

When we used the SCA with liquid, the sidewall was made of Lexan with an inner diameter $D = 9.53$ cm and a height $L = 9.96$ cm (resulting in $\Gamma = 0.96$).

We used the medium convection apparatus (MCA) of Refs. \cite{22,47,59} for measurements with liquids. Prandtl number values of 23.9, 28.9, and 35.6 were studied using isopropanol at average temperatures $T_m = 50$ °C, 40 °C, and 30 °C, respectively. In these experiments the height and the diameter of the convection cell were $D = L = 24.8$ cm. We also used the fluorocarbon $C_2F_4$ (3M Fluorinet FC72) which yielded an intermediate $Pr = 12.3$ (for more information on that fluid see Ref. \cite{22}). For these measurements $T_m = 25$ °C. Two samples, both with $\Gamma = 1.00$ but of different physical size, were used. The smaller one, to be designated MCA-S, had $L = D = 19.0$ cm, while the dimensions for the larger one (MCA-L) were $L = D = 24.1$ cm. The sidewalls for the MCA-S and MCA-L were made of Lexan and acrylic, respectively. The top and bottom plates of both cells were made of copper.

Either two (SCA) or four (MCA) thermistors were placed inside the bottom plate close to its upper surface for temperature measurements. The thermistors were calibrated against a calibrated platinum thermometer in a separate apparatus before their installation in the plates. Metal-film electrical heaters were attached at the undersides of the bottom plates. The heater power was measured using a four-lead method. The copper top plate of the MCA cell had an integrated cooling system through which a cooling liquid (ethylene glycol) circulated. The cooling-liquid temperature was set by a temperature-controlled chiller (NESLAB), such that the top plate was at the desired temperature $T_t$. The sapphire top plate of the SCA cell was cooled by water circulating over its top surface. The water temperature (and thus $T_t$) was controlled using a digital feedback loop. A small correction was applied for the temperature drop across the sapphire plate.

The rotating table and an outer structure supporting electrical and liquid feed-throughs from the laboratory to the rotating frame were similar for the SCA and the MCA, as described in detail in an earlier publication \cite{47}.

With the liquids used in the present work and various $T_m$ the range $0.74 \lesssim Pr \lesssim 35.5$ could be covered. The Rayleigh numbers ranged from $3.6 \times 10^4$ with $N_2$ at $Pr = 0.74$ to $4 \times 10^{11}$ with FC72 at $Pr = 12.3$. An overview of all $Pr$ and $Ra$ values used is shown in Table I.

For most of the experiments, we held $Ra$ and $Pr$ constant, while measuring the heat transport for several different rotation rates (i.e., different $Ro$ or $Ek$). In Fig. 2, we show all runs in the $Ro-Ra$ and the $Ro-Ek$ parameter space. Nu data for water ($Pr = 3.05, 4.38$, and 6.26) were published before in Ref. \cite{47} and are included in the figure and in our analysis.
TABLE I. Overview of the control parameters realized in the experiments.

<table>
<thead>
<tr>
<th>Apparatus</th>
<th>L (cm)</th>
<th>Fluid</th>
<th>Pr</th>
<th>Ra</th>
<th>Nu(1/Ro = 0)</th>
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<tbody>
<tr>
<td>SCA</td>
<td>9.8</td>
<td>N₂</td>
<td>0.74</td>
<td>3.6 × 10⁸</td>
<td>49.9</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
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<td>4.6 × 10¹⁰</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>9.6</td>
<td>FC₇₂</td>
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V. RESULTS

A. Nusselt-number measurements

In this section we present new measurements of Nu(Ra, Pr, 1/Ro) before proceeding, in Sec. VB, to a global analysis of all available data.

1. Nuᵣ for FC₇₂ and Isopropanol—large Pr

Figures 3 and 4 display results for Nuᵣ as a function of 1/Ro for large Pr. The data were taken with FC₇₂ (Fig. 3, Pr = 12.3) and isopropanol (Fig. 4, Pr = 23.9, 28.9, and 35.6). The results for Pr = 12.3 are qualitatively consistent with those for water (Pr = 4.38) shown in Fig. 1 (compare also with results for other Ra and nearby Pr [47]) in that they reveal the three heat-transport regimes. However, they extend to larger Ra by more than a decade. The different Nuᵣ behavior in the different 1/Ro ranges is clear. After a range I, where Nuᵣ does not change much, Nuᵣ decreases (Iᵣ) and reaches a minimum at 1/Roᵣ. As seen for the water data at smaller Pr, the transition from range I to range II at 1/Roᵣ is, within the resolution of the data, independent of Ra. For 1/Ro > 1/Roᵣ, Nuᵣ increases again in range II. While the range of 1/Ro covered by the data was not sufficient to reach a maximum of Nuᵣ for

FIG. 2. Overview of all data. The top plot shows the Ro-Ra parameter space, and the bottom plot shows the same points in the Ro-Ek parameter space. The data for Pr = 3.05, 3.62, 4.38, 6.26, and 6.41 were published already in Ref. [47] but are included in the present analysis.

Ra = 4.16 × 10¹⁰, data at both smaller and larger Ra did show such a maximum Nuᵣ,max at 1/Roᵣ,max after which, in range III, Nuᵣ decreased again. One sees that the largest value of both Nuᵣ,max and 1/Roᵣ,max is reached for the smallest Ra. These trends are qualitatively consistent with data obtained using water.

Especially the Nuᵣ data for the two largest Ra show further rather sharp transitions above 1/Roᵣ. While the detailed nature of the states above and below these transitions still needs to be elucidated, this phenomenon was discussed at length in a previous publication [39].

The trend of a diminishing Nuᵣ,max with increasing Ra of the data for Pr = 12.3 that is shown in Fig. 3 is consistent with previous results for Pr = 5.9, Ra = 4.3 × 10¹⁵ [62] which revealed a slow decrease of Nuᵣ with increasing 1/Ro, without any evidence of a maximum.

Experimental results for isopropanol (Pr = 23.9, 28.9, and 35.5) are shown in Fig. 4. There is a larger heat-transport enhancement for these larger values of Pr than was seen at smaller Pr. None of the data extend to 1/Roᵣ,max where Nuᵣ reaches Nuᵣ,max. Nevertheless, for the smallest Ra heat-transport enhancement of up to 40% was reached, and thus Nuᵣ,max is even larger than 1.4.

In general, these data are qualitatively similar to the water data in Ref. [47]. For larger Ra there is a 1/Ro range where Nuᵣ decreases until it reaches a minimum (range I), followed by a 1/Ro range where Nuᵣ increases (range II). Again, we could not observe the maximal Nuᵣ and the change to range III, but we see that the slope Sᵣo [Eq. (2)] decreases with
FIG. 3. Relative Nusselt numbers $N_{Ur}$ for measurements at $Pr = 12.34$ (FC72) and $Ra = 1.91 \times 10^{10}$ (squares, red online), $1.00 \times 10^{11}$ (diamonds, green online), $2.07 \times 10^{11}$ (up-pointing triangles, black online), and $4.24 \times 10^{11}$ (down-pointing triangles, purple online). The right plot shows the same data as the left one, but zoomed closer to the onset of heat-transport enhancement. The uncertainty of the data is represented by their scatter and decreases with increasing $\Delta T$ (i.e., increasing $Ra$). For the three largest $Ra$, the uncertainty is smaller than the symbol size. Open symbols mark predictions of the maximum heat transport, as explained in Sec. VB4. Data for $Ra = 2.07 \times 10^{11}$ and $1.00 \times 10^{11}$ were published already in Ref. [39].

FIG. 4. The reduced Nusselt number $N_{Ur}$ for isopropanol and $Pr = 23.9$ (a) and (b), 28.9 (c) and (d), and 35.5 (e) and (f). The different symbols stand for $Ra = 2.2 \times 10^9$ (bullets, blue online), $9.0 \times 10^9$ (squares, red online), and $1.8 \times 10^{10}$ (diamonds, green online). The left column [(a), (c), and (e)] covers a wide range of $1/Ro$ on a logarithmic scale, while the right column [(b), (d), and (f)] gives more detail at smaller $1/Ro$ on a linear scale. The uncertainty of the data is represented by the scatter of the data points. Open symbols in (a), (c), and (e) mark predictions of $N_{Ur,\text{max}}$ as explained in Sec. VB4.
The left column shows the whole available 1/Ro range on a logarithmic scale in order to focus more closely on the range. In this context it is also noteworthy that, for the smallest Pr, where Ekman pumping is too feeble to be noticeable in the heat transport [46,52], the formation of Ekman vortices from plumes still takes place when a critical value of 1/Ro has been exceeded. In this parameter range one then would have to identify the transition on the basis of the 1/Ro dependence of other properties, such as internal temperature gradients [39] or other local measurements.

B. Dependence of Nu, on Pr and Ra

1. Qualitative trends

In this section we examine all available data in order to learn about the Pr and Ra dependence of Nu, . To this end we show in Fig. 6 three plots, each for a given Ra, of Nu, as a function of 1/Ro for different Pr. The plots are for Ra = 2 × 10^9 (top row), 9 × 10^9 (middle), and 18 × 10^9 (bottom). The left column shows the whole available 1/Ro range on a logarithmic x axis. The right column shows only the range 1/Ro ≤ 2 on a linear scale in order to focus more closely on 1/Ro, where the heat-transport enhancement sets in. The maximal enhancement Nu,max of Nu, is seen most clearly by eye in the left column. As already apparent from Fig. 4, the largest enhancement occurs for the two largest Pr of 35.5 (brown up-triangles) and 28.9 (black down-triangles). For these Pr values the rotation rates 1/Romax at which the heat transport reaches its maximum could not be reached in the experiment, but the available data already show an increase of Nu, by up to 40%. In Fig. 6(a), there are only two maxima visible (for Pr = 6.26 and 4.38). It becomes apparent that Nu,max, as well as its location 1/Romax, increase with Pr. The same trend is also shown for larger Ra in Figs. 6(c) and 6(e), although there Nu,max is smaller. These results are in general agreement with numerical simulations [52]. Also in agreement is the fact that, for the very low Pr ≈ 1, there is no or only a very small Nu enhancement as discussed above in Sec. VA2.

Looking closer at Fig. 6(a), one also sees that there is a range 0.4 < 1/Ro < 3 over which the slope S_10 of Nu, (1/Ro) is independent of or at most weakly dependent on Pr. Over that range all data points within their scatter collapse onto a single curve. For larger 1/Ro the data diverge from each other. For Pr ≤ 6.26 they reach their maxima and decrease afterwards, while for larger Pr they continue to increase over our 1/Ro range. The seemingly equal slope just above the onset of heat-transport enhancement can also be seen for larger Ra in Figs. 6(c) and 5(e). However, we shall show below that the initial slope does depend, albeit only weakly, on both Ra and Pr when analyzed in more detail and over a wider parameter range.

In this context it is also noteworthy that, for the smallest Ra (Ra = 2 × 10^9) in Fig. 6 and near our largest 1/Ro, Nu, near our largest Pr decreases with increasing Pr [see brown and black triangles in Fig. 6(a)]. A similar behavior was found in numerical simulations [52] (see, for instance, Fig. 2 of that observed at this position. Noteworthy is the fact that the initial increase and the maximum of Nu, seen for small Ra occurs before the estimated 1/Romax. This implies that the enhancement of less than 2% is not caused by Ekman pumping, but rather by another still unknown mechanism. We note that the absence of any signature in Nu, at the expected location of 1/Romax does not imply the absence of a transition. Presumably even for small Pr, where Ekman pumping is too feeble to be noticeable in the heat transport [46,52], the formation of Ekman vortices from plumes still takes place when a critical value of 1/Ro has been exceeded. In this parameter range one then would have to identify the transition on the basis of the 1/Ro dependence of other properties, such as internal temperature gradients [39] or other local measurements.

FIG. 5. Reduced Nusselt number for compressed gases. Data show Nu, as a function of 1/Ro for Ra = 3.6 × 10^8 (bullets, blue online), Ra = 9 × 10^8 (squares, red online), Ra = 1.8 × 10^9 (diamond, green online), Ra = 2.3 × 10^10 (open circles, brown online), and Ra = 4.6 × 10^10 (down triangle, orange online). Data for the three lowest Ra were acquired using nitrogen (Pr = 0.76), the two data sets for the larger Ra were acquired using SF6 (Pr = 0.84). We also show data from numerical simulations at Ra = 10^9 (up-triangles, black online) from Ref. [46]. The uncertainty of the data is represented by only one point, the authors of Ref. [52] also show data from numerical simulations at Ra = 10^10 (open circles, brown online), and Ra = 10^11 (top row), 9 × 10^9 (middle), and 18 × 10^9 (bottom).

2. Nu, for N_2 and SF_6—Pr near one

We show in Fig. 5 the data for Nu, as a function of 1/Ro for Pr = 0.74 and Pr = 0.84. One sees that the heat-transport enhancement is quite small or absent. For the smallest Ra (Ra = 3.6 × 10^8) the increase of Nu, at its maximum is only 2%. A very small enhancement was observed also in numerical simulations for Ra = 2 × 10^8, albeit for a cylindrical sample with Γ = 0.5 [63]. Simulations for Γ = 1, Pr = 0.7, and Ra = 10^8 [52] are shown in Fig. 5 as black triangles. They agree with our experimental data fairly well. Even a slight enhancement of about 1.5% is visible. However, since the enhancement is represented by only one point, the authors of Ref. [52] apparently considered it an outlier since they stated “...no heat transport enhancement is found for Pr = 0.7...”. Similar to experiments at larger Pr, the heat-transport enhancement is larger for smaller Ra. For Ra ≥ 1.8 × 10^9, no enhancement can be found.

The exact reason for the strongly reduced heat-transport enhancement at lower Pr is not known. It was argued that the larger heat diffusion makes Ekman pumping less efficient for smaller Pr [46,52]. Warm fluid that is transported by Ekman pumping across the bottom boundary loses its heat in the horizontal direction more quickly. While this may enhance the vertical temperature gradient in the bulk, it reduces the gradient in or near the boundary layers. We also note, that small Pr indicates small Ekman-layer thicknesses δ_E. We also show in Fig. 5 the location of the expected 1/Ro, (dashed vertical line), as estimated from the power-law relation between Pr and 1/Ro, (see Sec. V B). No significant change is
HEAT-TRANSPORT ENHANCEMENT IN ROTATING . . . PHYSICAL REVIEW E 93, 043102 (2016)

Fig. 6. Comparison of Nu_r as a function of 1/Ro for different Pr and Ra = 2 × 10^9 (a) and (b), 9 × 10^9 (c) and (d), and 2 × 10^10 (e) and (f). The left column shows the full experimentally accessible range on a logarithmically scaled x axis. The right column shows the range 0 < 1/Ro < 2 using a linear x axis. Different symbols stand for different Pr as explained at the top of the figure. The horizontal black dashed line marks Nu_r = 1. Note that different vertical scales were used because of the different ranges of the data.

In the following sections we analyze the data sets more quantitatively and investigate how the (i) onset of heat-transport enhancement at 1/Ro_c, (ii) the initial slope S_Ro, and (iii) the maximal heat-transport enhancement Nu_r,max change with Ra and Pr.

2. The onset of heat-transport enhancement

Heat-transport enhancement due to Ekman pumping sets in at a particular 1/Ro_c, causing a sharp transition with a discontinuous slope S_Ro. There is a small decrease of Nu_r right before 1/Ro_c (range Ib), leading to an effective reduction of the heat transport and a local minimum Nu_r,min at 1/Ro_c.

In previous papers [36,37] it was argued that a sharp onset at a finite 1/Ro_c is due to the finite lateral dimension of the cylinder. Experimental measurements of 1/Ro_c for several aspect ratios Γ = D/L indicated that 1/Ro_c → 0 as Γ → ∞ (i.e., as the diameter D → ∞). A Ginzburg-Landau-like equation [64] for the vortex density in a plane in the bulk but close to the thermal boundary layers, with appropriate boundary conditions, modeled the phenomenon as a finite-size effect. Consistent with this interpretation of the experimental results, simulations for the same parameters as the experimental measurements [36] found that the vortex density increases with 1/Ro when 1/Ro ≥ 1/Ro_c. Recent vortex-visualization experiments also showed that the vortex density has a maximum close to the side wall (although it vanishes at the wall) when 1/Ro is not too large, suggesting that vortices tend to repel each other [50].

reference where Nu_r decreases with increasing Pr for 1/Ro = 10 and Pr ≳ 15). There, they found for various constant 1/Ro and Ra, optimal Pr for which Nu_r was maximal. A direct comparison with the numerical results is not possible, however, because the DNS was done at smaller Ra (Ra = 10^9) than the experiment.
In order to illustrate the transition at 1/\( \text{Ro}_c \), we plot in Figs. 6(b), 6(d), and 6(f) \( \text{Nu}_r \) over a limited range of 1/\( \text{Ro} \) on linear scales. The data for \( \text{Ra} = 2 \times 10^9 \) scatter significantly for small 1/\( \text{Ro} \) and thus a precise value of 1/\( \text{Ro}_c \) cannot be obtained. However, for \( \text{Ra} = 2 \times 10^{10} \) [Fig. 6(f)] the onset at 1/\( \text{Ro}_c \) is clearly visible. Here we see that 1/\( \text{Ro}_c \) depends on \( \text{Pr} \), being smaller for larger \( \text{Pr} \).

To quantify the location of 1/\( \text{Ro}_c \), we fit the function,

\[
\text{Nu}_r = \begin{cases} 
\frac{S^-_\text{Ro}}{\text{Ro}} + n; & 1/\text{Ro} < 1/\text{Ro}_c \\
\frac{S^+_\text{Ro}}{\text{Ro}} + (S^-_\text{Ro} - S^+_\text{Ro})/\text{Ro}_c + n; & 1/\text{Ro} > 1/\text{Ro}_c,
\end{cases}
\]

(11)

to the data points close to the onset. In this way we get not only 1/\( \text{Ro}_c \), but also the slopes \( S^-_\text{Ro} \) and \( S^+_\text{Ro} \) just below and above 1/\( \text{Ro}_c \) and the minimal relative Nusselt number \( \text{Nu}_{r,\text{min}} = S^-_\text{Ro}/\text{Ro}_c + n \) which occurs at 1/\( \text{Ro}_c \). [65].

In Fig. 7(a) we show 1/\( \text{Ro}_c \) as a function of \( \text{Pr} \) on double-logarithmic scales. The data can be represented by a straight line and a fit yielded the power law,

\[
1/\text{Ro}_c = K_1 \text{Pr}^{\alpha_1},
\]

(12)

with an exponent \( \alpha = -0.41 \pm 0.02 \) and a coefficient \( K_1 = 0.75 \pm 0.02 \). A very similar behavior with nearly the same exponent (within its margin of error) was already found before [47]. There, however, only data acquired with water as the convective fluid were used, spanning the rather short \( \text{Pr} \) interval of 3.05 \( \leq \text{Pr} \leq 6.26 \). To our knowledge the mechanism that yields the \( \text{Pr} \) dependence of 1/\( \text{Ro}_c \) remains to be elucidated.

From Fig. 6(b) it appears that the influence of \( \text{Ra} \) on the location of 1/\( \text{Ro}_c \) is very weak or absent. In order to search for a small effect, we plot in Fig. 7(b) the reduced form \( \text{Pr}^{0.41}/\text{Ro}_c \) as a function of \( \text{Ra} \) so as to remove the effect of \( \text{Pr} \). We see no trend in these data. Thus, within the resolution of the data \( \text{Ra} \) has no influence on the location of 1/\( \text{Ro}_c \) in the range \( 10^6 \leq \text{Ra} < 10^{12} \), and the coefficient \( K_1 \) in Eq. (12) is within experimental uncertainty independent of \( \text{Ra} \). We note that Eq. (7) then implies that the transition value of the inverse Ekman number 1/\( \text{Ek} \) is proportional to \( \text{Ra}^{0.50} \) and \( \text{Pr}^{\alpha_1-1/2} = \text{Pr}^{-0.91} \).

3. The slopes before and after 1/\( \text{Ro}_c \)

By fitting Eq. (11) to the data for \( \text{Nu}_r/(1/\text{Ro}) \) near 1/\( \text{Ro}_c \), we also obtained the slopes \( S^-_\text{Ro}, S^+\text{Ro} \), and the reduced Nusselt number \( \text{Nu}_{r,\text{min}} \) at the onset.

Zhong and Ahlers [47] show plots of \( S^+_\text{Ro} \) as a function of \( \text{Ra} \) and \( \text{Pr} \) (see Fig. 7 of Ref. [47]). They found no dependence of \( S^-_\text{Ro} \) on \( \text{Ra} \), but a seemingly significant increase with \( \text{Pr} \) corresponding to a power law with an exponent of 0.27. However, their data covered only a small range of \( \text{Pr} \) and an accurate determination of the \( \text{Pr} \) dependence thus was not possible. We find that plotting the much more extensive data for \( S^-_\text{Ro} \) that are now available as a function of \( \text{Ra} \) or \( \text{Pr} \) as was done in Ref. [47] does not reveal an obvious \( \text{Ra} \), or \( \text{Pr} \) dependence of \( S^-_\text{Ro} \). A monotonic increase of \( S^-_\text{Ro} \) with \( \text{Pr} \) as suggested in Ref. [47] is not observed.

Until now, we considered the heat-transport enhancement as a function of the inverse Rossby number 1/\( \text{Ro} \). This is useful since 1/\( \text{Ro} \) is of order unity in the range where the heat transport is enhanced, and in retrospect because it leads to a \( \text{Ra} \)-independent transition. We could have also considered the inverse Ekman number 1/\( \text{Ek} \) as the control parameter (top x axis in Fig. 1). As mentioned above, the scaling of the critical inverse Ekman number at which heat-transport enhancement sets in (1/\( \text{Ek}_c \)) would then be different. In addition, the slope \( S^+_\text{Ek} \) [see Eq. (9)] of \( \text{Nu}_r/(1/\text{Ek}) \) would also be different. From Eq. (7) one sees that \( S^+_\text{Ek} = 2\sqrt{\text{Ro} \text{Pr}}/\text{Ra} \). \( S^+_\text{Ek} \) has the advantage that it covers almost two orders of magnitude for our data sets, whereas \( S^-_\text{Ro} \) changes only by a factor of two. In Fig. 8(a) we plot \( S^+_\text{Ek} \) as a function of \( \text{Ro} \). One sees that \( S^+_\text{Ek} \) decreases monotonically with increasing \( \text{Ra} \). In this double-logarithmic plot a power-law behavior \( S^+_\text{Ek} \propto \text{Ra}^\beta \) is visible with an exponent \( \beta \approx -0.60 \). The data points in Fig. 8(a) are color coded with respect to their corresponding \( \text{Pr} \). We see that greenish points (large \( \text{Pr} \)) lie in general above the fitted straight line while reddish points (low \( \text{Pr} \)), are below, suggesting that \( S^+_\text{Ek} \) also depends on \( \text{Pr} \). In order to investigate the \( \text{Pr} \) dependence, we plot in Fig. 8(b) the reduced slope \( S^+_\text{Ek} \text{Ra}^{0.60} \) versus \( \text{Pr} \). This quantity increases with \( \text{Pr} \), showing a power-law dependence with an exponent of about -0.31.

While these two exponents are not yet the best-fit values, they are good starting parameters for a two-dimensional nonlinear least-square fit of the equation,

\[
S^+_\text{Ek} = K_\text{Ek} \text{Pr}^{\beta_1} \text{Ra}^{\beta_2},
\]

(13)

to the data. This fit yielded \( \beta_1 = 0.34 \pm 0.05 \) and \( \beta_2 = -0.54 \pm 0.08 \). Having both exponents, we plot in Fig. 9(a) \( S^+_\text{Ek} \) as a function of \( \text{Pr}^{\beta_1} \text{Ra}^{\beta_2} \). We see that all data points lie on
a straight line. A linear fit gives a slope of $K_{Ek} = 0.5 \pm 0.02$ and a negligible intercept of $n < 10^{-6}$. From this consideration we calculate that $S^+_{Ro}$ can similarly be expressed as

$$S^+_{Ro} = K_{Ro} Pr^{\beta_1} Ra^{\beta_2},$$  

(14)

with $\beta_1 = -0.16 \pm 0.05$ and $\beta_2 = -0.04 \pm 0.08$. Figure 9(b) shows $S^+_{Ro}$ plotted as a function of $Pr^{0.16}Ra^{-0.04}$. Again, the data points follow a straight line, in this case with a slope $K_{Ro} = 0.3 \pm 0.05$. The scatter seems larger than it is for $S^+_{Ek}$; however, this is merely due to the rather small range of $S^+_{Ro}$ which also leads to a larger uncertainty of $K_{Ro}$.

We note that the results for $\beta_1$ and $\beta_2$ differ from those found by analyzing only the water data as done in Ref. [47] and which yielded $S^+_{Ro} = 0.058Pr^{0.27}$ (no dependency of $Ra$). This difference is because we (i) used a significantly larger data set, (ii) analyzed the data in a truly two-dimensional way that takes $Ra$ and $Pr$ dependency into account, and (iii) analyzed the data first in terms of $1/Ek$ and later converted $S^+_{Ek}$ to $S^+_{Ro}$.

The slope right before $1/Ro_c$ ($S_{Ro}$) is shown in Fig. 10. Even though the data scatter significantly, one can see that $S_{Ro}$ decreases with increasing $Ra$ and increasing $Pr$. While $S_{Ro}$ is positive for the smallest $Ra$ and $Pr$ (red bullets), it becomes slightly negative as $Ra$ or $Pr$ increases. However, a quantitative analysis is difficult and in particular a separation of the $Ra$ and $Pr$ dependencies was not possible.

Another parameter obtained from fitting Eq. (11) to the data is the Nusselt number at $1/Ro_c$ ($Nu_{r, min}$). In order to investigate
to the data points close to $1/R_{\text{Ro}}$, adjusting $a, x_0, b,$ and $c$. Examples of such fits are shown in Fig. 1. Over a wide range of $Ra$ this function represents the data remarkably well. It takes the asymmetry of the peak into consideration while assuring the existence of only a single maximum (in contrast to a third-order polynomial, for instance). In this way, we can determine $1/R_{\text{Ro}}$ and $Nu_{r, max}$ with better accuracy than by just using the data points with the largest $Nu_r$. We appreciate that this procedure is questionable at very large $Ra$ where the discontinuities of $S_{\text{Ro}}$ are dominating the shape of $Nu_r(1/Ro)$.

Since we are mostly interested in the additional heat transported due to rotation, we focus on $Nu_{r, max} - 1$. It would be convenient to find a simple power-law representation,

$$Nu_{r, max} - 1 = M_{\text{Ro}} Pr^{\delta_1} Ra^{\delta_2},$$

(16)
of the data. However, plotting $Nu_{r, max} - 1$ as a function of either $Ra$ or $Pr$ for all data points does not show a significant trend in either of these plots (not shown here), and an immediate two-dimensional least-squares fit of Eq. (16) to the data was not successful without a good guess at the initial values of the parameters. Thus, we take an approach similar to that of Sec. VB3, i.e., we first plot $Nu_{r, max} - 1$ as a function of $Pr$ and fit the power law $Nu_{r, max} - 1 = a_1 Pr^{\delta_1}$ to it. The value of $\delta_1$ is then used to create a reduced form $(Nu_{r, max} - 1)Pr^{-\delta_1}$, which we plot as a function of $Ra$. A similar fit of $(Nu_{r, max} - 1)Pr^{-\delta_1} = a_2 Ra^{\delta_2}$ gives the exponent $\delta_2$ that we again use to plot the first reduced form $(Nu_{r, max} - 1)Ra^{-\delta_1}$ vs $Pr$ and so on. In this iterative way, the exponents converge quickly (after four iterations) to $\delta_1 = 0.65, \delta_2 = -0.35$.

Figure 12 shows the results of the last iteration. One sees that a power-law dependence on $Pr$ and $Ra$ agrees fairly well with the data. However, while the data cover two orders of magnitude of $Ra$, they span less than a single order of magnitude of $Pr$. In addition one sees in Fig. 12(a) that points for small $Pr$ and for the largest $Pr$ are below the fitted line, whereas points in between are above it. Because of this, $\delta_1$ is sensitive to the data used in the fit. Removing the point at the highest $Pr = 23.9$ increases $\delta_1$ by 20%, while $\delta_2$ changes by only 5% change.

The above procedure gave excellent initial values for a direct two-dimensional fit of Eq. (16) to the data. That fit yielded $M_{\text{Ro}} = 42 \pm 0.27, \delta_1 = 0.80 \pm 0.08$, and $\delta_2 = -0.35 \pm 0.03$. Again, while the $Pr$ exponent differs, the $Ra$ exponent stays unchanged. The reason for the different results from the two methods is that data points in the two approaches are weighted differently. In the least-squares-fit method, every data point has equal weight. The single point at $Pr$ = 23.9 has little influence since there is a large cluster of points between $Pr = 3$ and 6.3. In the iterative approach the point at $Pr = 23.9$ gets more weight and thus the exponent decreases significantly.

We show in Fig. 13 plots of $Nu_{\text{max}}$ as a function of both, $Pr^{\delta_1} Ra^{\delta_2}$ (blue bullets) and $Pr^{\delta_1} Ra^{\delta_2}$ (red squares). As expected, the points follow their corresponding linear master curves. The slopes are found to be $M_{\text{Ro}} = 64 \pm 2$ and $M_{\text{Ro}} = 44 \pm 2$, respectively.

The situation for $1/R_{\text{Ro}}$ is similar. We applied a similar iterative method as above to reveal a potential power-law

\[ f(x) = a \log(x - x_0) + bx + c; \quad x = 1/Ro, \]

(15)
The largest $Ra$ ($1.9 \times 10^{10}$) lies significantly below the fitted line. While this might be evidence that the $Ra$ dependence is more complicated than just a simple power law, the error of this data point is also rather large; i.e., there is a large uncertainty of $1/Ro_{\max}$.

A direct two-dimensional fit, using the exponents $\epsilon_1$ and $\epsilon_2$ as starting values, gives very similar exponents $\tilde{\epsilon}_1 = 1.37 \pm 0.05$ and $\tilde{\epsilon}_2 = -0.18 \pm 0.03$. 

FIG. 12. (a) Rescaled heat-transport enhancement $(Nu_{r,\max} - 1)Ra^{0.35}$ as a function of $Pr$. Color of the symbols stands for the log$_{10}(Ra)$. (b) Rescaled heat-transport enhancement $(Nu_{r,\max} - 1)Pr^{-0.65}$ as a function of $Ra$. Color of each symbol represents $Pr$. 

FIG. 13. The maximal heat-transport enhancement $Nu_{r,\max} - 1$ as a function of $Pr^{0.5}Ra^{0.5}$ (blue bullets) and $Pr^{0.5}Ra^{0.5}$ (red squares). The blue solid line and the red dashed line are linear fits to the data points with slopes $M_{Ro} = 64$ and $\tilde{M}_{Ro} = 44$.

FIG. 14. (a) Rayleigh-reduced inverse Rossby number $Ra^{0.17}/Ro_{\max}$ at the maximum of $Nu_{r}$ as a function of $Pr$. The decimal logarithm of the corresponding $Ra$ is color coded. (b) Prandtl-reduced $Pr^{-1.45}/Ro_{\max}$ as a function of $Ra$. The corresponding $Pr$ is color coded. In both plots, the axes are logarithmically scaled.

FIG. 15. Location of the maximal heat-transport enhancement $1/Ro_{\max}$ as a function of $Pr^{1.37}Ra^{-0.18}$. The data follow a linear trend (red solid line) with slope $N_{Ro} = 21.4 \pm 0.5$. 

relationship of the form,

$$1/Ro_{\max} = N_{Ro}Pr^{\epsilon_1}Ra^{\epsilon_2}, \quad (17)$$

with $\epsilon_1 = 1.45 \pm 0.06$ and $\epsilon_2 = -0.17 \pm 0.03$. This result is shown in Fig. 14 where the corresponding reduced forms of $1/Ro_{\max}$ are plotted as a function of $Pr$ and $Ra$. One sees in Fig. 14(a), that $Ra^{\epsilon_2}/Ro_{\max}$ follows nicely a power-law dependence on $Pr$. The reduced $Pr^{-1-\epsilon_1}/Ro_{\max}$ as a function of $Ra$ is plotted in Fig. 14(b). There the data also seem to follow a power law, although the data scatter significantly more and deviate more from the best fit. Especially the data point with
In Fig. 15 we plot a master curve of $1/Ro_{\text{max}}$ as a function of $Pr^{1.37}Ra^{-0.18}$. All data points follow a straight line with a slope $N_{Ro} = 21.4 \pm 0.5$. One sees that the assumption of a power-law dependence is a good approximation. Even the single point at the right top corner still agrees with the fitted curve. We note, however, that that particular $1/Ro_{\text{max}}$ value is uncertain because it was obtained by an extrapolation of Eq. (15) beyond the data used in the fit; the maximum lies beyond the experimentally accessible $1/Ro$ range as seen in Fig. 4(a).

Using the calculated exponents $\delta_1 = 0.65$, $\delta_2 = -0.35$, $\varepsilon_1 = 1.37$, and $\varepsilon_2 = -0.18$ and the amplitudes $M_{Ro} = 64$ and $N_{Ro} = 21.4$, we can make predictions of $Nu_{r,\text{max}}$ and $1/Ro_{\text{max}}$ for the data sets for which we could not reach $1/Ro_{\text{max}}$ in the experiment. These predictions are plotted in Figs. 3 and 4 as open symbols. Let us consider first the large-$Pr$ case (Fig. 4). In almost all cases the predicted $Nu_{r,\text{max}}$ is reasonable, with the only exception for $Pr = 28.9$ and $Ra = 2.2 \times 10^9$ where the predicted value is actually smaller than the largest measured value. For the smaller $Pr = 12.34$ we see that the prediction is convincing only for $Ra = 4.16 \times 10^{10}$. It is clearly in error for $Ra \gtrsim 10^{11}$. It is noteworthy that this is the range where sharp transitions between states, involving large changes of the slope $S_{Ro}$, are found; these features clearly are not captured by Eq. (15).

VI. SUMMARY AND DISCUSSION

A. New experimental data

In this paper we report on the heat-transport enhancement due to rotation in buoyancy-dominated turbulent thermal convection as a function of the Rayleigh number and the Prandtl number for samples with aspect ratio $\Gamma = 1.0$. We added experimental data for small Pr by making measurements with compressed gasses ($N_2$ with $Pr = 0.74$ and $SF_6$ with $Pr = 0.84$), for medium $Pr$ but exceptionally large $Ra$ by using the fluorocarbon FC72 ($Pr = 12.3$), and for large $Pr$ by using isopropanol ($Pr = 23.9, 28.9,$ and $35.5$). Together with previous data acquired with water [47] ($3.0 \lesssim Pr \lesssim 6.4$), all available measurements cover the Rayleigh-number range $4 \times 10^8 < Ra < 4 \times 10^{11}$ and values of $Pr$ from 0.74 to 35.5. This large data set makes it possible for the first time to study quantitatively the influence of $Pr$ on the heat-transport enhancement in rotating turbulent Rayleigh-Bénard convection.

B. Three different ranges of $1/Ro$

It was shown already [37,39,47,58] that at least three different ranges of the rotation rate $1/Ro$ can be observed, with qualitatively different slopes $S_{Ro}$ [see Eq. (2)] in each of them. These different ranges are illustrated in Fig. 1. Starting at small $1/Ro$, the first (I) is one of nearly constant heat transport $Nu_r$. For large enough $Ra$ it can be separated into two subranges. In the first ($I_a$) $Nu_r$ increase very slightly, by a few tenths of a percent. It is followed by subrange $I_b$ where $Nu_r$ decreases by a percent or so. In ranges $I_a$ and $I_b$ the turbulent flow is self-organized into a large-scale circulation consisting of a single convection roll (LSC). The stochastic dynamics, amplitude, and stability of the LSC are influenced by Coriolis forces as indicated by previous measurements [47,48]. Although one might have thought that the elucidation of phenomena at these smallest rotation rates should be most accessible to theory, the origin of the two subranges $I_a$ and $I_b$, and the dependence of $Nu_r$ upon $Ra$ in these ranges, remain unexplained at present.

Range I is followed by a range II where the heat transport increases up to a rather broad maximum. Ranges I and II are separated by a sharp transition at $1/Ro_a$. The transition marks the onset of Ekman-vortex formation due to the action of the Coriolis force on thermal plumes emitted from the boundary layers. Remarkably, in samples of finite lateral extent the vortex formation can only start after a critical value $1/Ro_a$ is exceeded. This has been explained theoretically as a finite-size effect that can be described by a Ginzburg-Landau model [36,37]. The increase of $Nu_r$ above $1/Ro_a$ in range II is understood to be due to the vortices near the thermal boundary layers adjacent to the top and bottom plate; these vortices cause an increased fluid transport by means of Ekman pumping from the plates across the boundary layers into the bulk. In this range the LSC is no longer stable. Its geometry is incompatible with the vortex formation, and it soon ceases to exist for $1/Ro > 1/Ro_a$.

Range II is followed by range III of decreasing heat transport. Also ranges II and III divide into subranges for larger $Ra$, where $S_{Ro}$ has noticeable discontinuities at several values of $1/Ro$. The specific reasons for this subdivision need further elucidation, but are expected to be found in transitions between turbulent states with large-scale structures of different symmetries [39].

A gradual decrease with increasing $1/Ro$ in range II of the slope $S_{Ro}$, and eventually an actual heat-transport reduction ($S_{Ro} < 1$) in range III, are due to a reduced vertical fluid transport with increasing $1/Ro$, caused by the Taylor-Proudman (TP) effect. While this qualitative picture is clear, a quantitative understanding of the heat-transport enhancement due to Ekman pumping and its depression due to the TP effect is still lacking.

C. Results of the present data analysis

In the present paper we carried out a correlation of new as well as previously published data describing the transitions between ranges I and II and between II and III. Within the resolution of the data we found that the transition between I and II at $1/Ro_a$ is independent of $Ra$ and decreases with $Pr$ (see Fig. 7). The decrease can be described by a power law with an exponent $\alpha = -0.41$ [Eq. (12)]. It is very surprising that $1/Ro_a$ is independent of $Ra$ over three decades. That means that the onset of the formation of vortices close to the boundaries does not depend on the thermal driving. However, we note that an alternative choice for the dimensionless rotation rate, like the inverse Ekman or the Taylor number, for example, would cause a $Ra$ dependence of the corresponding critical values $1/E_k$ [see Eq. (7)] and $T_a$. To our knowledge there is as yet no theoretical explanation of the $Pr$ dependence and the $Ra$ independence of $1/Ro_a$.

We determined the initial slope $S_{Ro}^*$ just above $1/Ro_a$ by fitting a linear function with a discontinuous derivative [Eq. (11)] to data points in ranges I and II but close to...
1/Ro. The initial slope $S^{+}_{Ro}$ just above 1/Ro was essentially independent of Ra and depended only weakly upon Pr (see Fig. 9). This weak dependence, and the limited data range especially for Pr, made it difficult to determine power-law exponents from a straightforward least-squares fit. It was easier to determine the Ra and Pr dependence of the slope $S^{+}_{Ek}$ derived from the dependence of Nu on the Ekman number Ek [Eq. (9)]. The results of this analysis could be transformed back to yield the power-law dependence $S^{+} \propto Pr^{\beta_1}Ra^{0.5}$ with $\beta_1 \approx -0.16$ and $\beta_2 \approx -0.04$. While at present these exponents remain unexplained, we expect that they may be central to a theoretical elucidation of Ekman pumping in this system.

We also investigated the location 1/Ro,max and the height $Nu_{r,max}$ of the maximum heat-transport enhancement. Analyzing these quantities was more difficult, since the maximum of Nu is broad and for large Pr was beyond the experimentally accessible range of 1/Ro. Therefore, the investigated Pr range is small and parameter uncertainties are relatively large (especially for 1/Ro,max). Also here we analyzed potential power-law dependencies of both quantities on Ra and Pr (see Figs. 12–14). We found that the data at constant Pr are consistent with $Nu_{r,max} \approx 1 \propto Ra^{-0.35}$. The Pr dependence at constant Ra could be given by $Nu_{r,max} \propto 1 \propto Pr^{0.55}$, although the error of this exponent is about 50%. We found that the location 1/Ro,max of Nu$_{r,max}$ can be represented by the power law $1/Ro_{max} \propto Pr^{0.37}Ra^{-0.18}$.

D. Concluding remarks

With our analysis we hope to provide sufficient data and a good starting point for theoretical modeling of rotating turbulent Rayleigh-Bénard convection in the buoyancy-dominated regime. Such modeling was already challenging for nonrotating convection where only two control parameters (Ra, Pr) are relevant [66–69]. It will be even more challenging for the rotating case where the parameter space is larger. However, the existence of the different 1/Ro ranges suggests that only a few mechanisms have to be considered, and that in each range a different mechanism is dominating the changes of the heat transport with increasing rotation rate. We note that analyzing 1/Ro, S^{+}_{Ro}, Nu$_{r,max} - 1$, and 1/Ro,max provides quantitative independent information about several mechanisms. From 1/Ro, we learn under which conditions vortices form close to the top and bottom boundaries (as shown in Ref. [36]) while $S^{+}_{Ro}$ is indicative of the strength of Ekman pumping once vortices have formed. The relative strength of Ekman pumping and TP suppression determines the values of Nu$_{r,max} - 1$ and 1/Ro,max.

Qualitatively one sees that $S^{+}_{Ro}$ is essentially independent of Ra and depends only weakly on Pr, indicating that Ekman pumping is not strongly Ra and Pr dependent. On the other hand, Nu$_{r,max} - 1$ and 1/Ro,max show a stronger Pr and Ra dependence, indicating that the location and size of the maximal heat-transport enhancement are determined mostly by the Ra and Pr dependencies of the heat-transport reduction in the bulk due to the TP effect. As 1/Ro increases through ranges II and III, the heat is still transported efficiently across the boundaries by Ekman pumping, but the thermal resistance of the bulk increases. As a result significant vertical temperature gradients develop in the bulk. Since the significance of the thermal resistance in the bulk increases with increasing 1/Ro and 1/Ek, and because 1/Ek = 1/(2Ro)\sqrt{Ra/Pr}, one would expect that at a given 1/Ro the thermal resistance increases with Ra but decreases with Pr. That is why, for constant Pr, Nu$_{r,max}$ is smaller at larger Ra and is reached already at smaller 1/Ro. On the other hand, for constant Ra and at larger Pr a larger Nu$_{r,max}$ is found and is reached only at larger 1/Ro.

In this paper we focused on the initial heat-transport enhancement Nu, with increasing 1/Ro and the decrease of Nu, for larger 1/Ro. We did not consider in detail the subranges Ia and Ib, and did not examine other subranges that occur for large Ra (see Sec. VA1 and Ref. [39]). These subranges become obvious in Nu$_r$(1/Ro) only at the largest Ra, but should be observable at smaller Ra by studies of internal flow structures.

Most likely transitions between structures of different symmetry in the turbulent bulk (such as, for instance, the replacement of Ekman vortices near the plates by Taylor columns penetrating the entire sample) and/or changes in the various boundary layers (thermal, Ekman, Stewardson) are responsible for the additional transitions above 1/Ro. Clearly, further experimental and numerical studies are needed for a better understanding of this very rich system.

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