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UNIVERSITY OF CALIFORNIA

Santa Barbara

Rayleigh-Bénard Convection in the Presence of a Weak Lateral Flow

A dissertation submitted in partial satisfaction of the requirements for
the degree of

Doctor of Philosophy

in

Physics

by

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June 1997
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1997
To my wife Gretchen,
for her constant support.
VITA

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Abstract

Rayleigh-Bénard Convection in the Presence of a Weak Lateral Flow

by

Steven P. Trainoff

Rayleigh-Bénard convection has, for several decades, been used as a paradigm for the study of pattern forming systems. It has been particularly fruitful since the fluid dynamical equations which underlie the flow are well known and because it can be well controlled experimentally. It is now routine to control the geometry and temperatures to better than 0.1% which allows for good comparison to theory. When one adds a lateral flow, the system demonstrates characteristics of open flow systems. In a large range of flows and temperature differences the system becomes convectively unstable and exhibits extreme sensitivity to fluctuations in the fluid. These fluctuations drive the observed pattern. This work will present measurements of the noise which drives the observed convection rolls. The magnitude of the noise is consistent with thermal fluctuations. This is an example of microscopic Brownian motion generating the macroscopic patterns which develop in the system.
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CHAPTER 1

Introduction

1.1 Introduction to the Introduction

1.1.1 What is Thermal Convection with Through Flow and Why is it Interesting?

Thermal convection is the familiar fluid motion which occurs when a layer of fluid is heated from below. It is observed every day when heating a pot of water on the stove, or when warming oil in a frying pan. The basic mechanism is that by heating from below, the fluid at the bottom becomes less dense than the fluid at the top (assuming the thermal expansion coefficient is positive). This gives rise to a buoyancy force which may destabilize the fluid layer leading to convective motion. 1 I say "may" because Lord Rayleigh showed in 1916 that there is a finite temperature difference below which there is no fluid motion at all [1]. This can be

---

1. The cellular convection one observes in thin layers of oil is actually driven by the temperature dependence of the surface tension, not buoyancy. In that sense I have unfortunately mislead the reader. It won't be the last time...
understood conceptually by considering the motion of a small element of fluid which is warmer than its surroundings. The fluid will have a net buoyancy force which will cause it to accelerate upwards. At the same time, viscosity acts to slow the motion and thermal diffusion causes the heat to flow out of the element. Rayleigh showed that this competition causes perturbations to be damped out when the temperature difference is below a critical value and grow when the temperature difference is above it. He showed that the system can be characterized by a non-dimensional number, now called the Rayleigh number, defined as

$$Ra = \frac{\alpha \Delta T d^3 g}{\kappa \nu}, \quad (1-1)$$

where $\alpha$ is the thermal expansion coefficient, $\Delta T = T_b - T_i$ is the temperature difference across the fluid layer, $d$ is the thickness of the layer, $g$ is the acceleration due to gravity, $\kappa$ is the thermal diffusivity, and $\nu$ is the kinematic viscosity of the fluid. The Rayleigh number is a dimensionless measure of the forces driving the convection. The numerator contains the quantities which enhance the convection, *i.e.* the buoyancy and the size of the cell, while the denominator contains the terms which suppress convection, *i.e.* the rate at which heat and momentum diffuse. Since the Rayleigh number depends linearly on the temperature difference across the cell, and the material parameters are only weak functions of the temperature, the Rayleigh number is a directly accessible control parameter. The
convection planform can potentially take may forms: rolls, hexagons, irregular cellular patterns, etc. It was shown by Schlüter, Lortz, and Busse in 1965 that when the fluid parameters are constant throughout the fluid layer, the pattern which appears at onset is a state of uniform rolls. At higher temperature differences, the uniform rolls have secondary instabilities to more complicated states. Eventually, the system becomes turbulent. This work will consider the dynamics of the rolls near the convective onset.

1.1.2 Absolute and Convective Instabilities

If one applies an external horizontal pressure gradient, the system acquires an extra control parameter in addition to the Rayleigh number. It is traditionally expressed in terms of the dimensionless Reynolds number which is defined by

$$\text{Re} = \frac{\bar{v}d}{\nu}, \quad (1-2)$$

where $\bar{v}$ is mean velocity in the channel. The experimental problem is illustrated in Figure 1-1. The system is a cell filled with fluid and bounded by isothermal plates on the top and bottom with temperatures $T_i$ and $T_b$, respectively. The sides of the cell have the same thermal conductivity as the fluid to avoid temperature gradients in $y$. The depth of

1. This is known as the Boussinesq approximation. See Section 1.2.1 for more details.
FIGURE 1-1. Cell Geometry

The cell is a long thin rectangular slab with a thickness of \(d\) and a width of \(\beta d\). For the purposes of discussion it will be assumed to be semi-infinite in length. There is an applied pressure gradient along the \(x\) direction which produces the through flow. Gravity is aligned along the negative \(z\) axis.

the cell is \(d\) and the width of the cell is \(\beta d\), where \(\beta\) is the lateral aspect ratio. For the purposes of this discussion the cell length \(l_d\) will be considered to be semi-infinite. The coordinate system is defined so that \(x\) is along the channel, \(y\) is across the channel, and \(z\) is vertical. The origin of the coordinate system is the center of the cell cross section at the inlet. The pressure gradient is applied in the negative \(x\) direction to cause a mean flow even in the absence of a temperature gradient.

The problem of thermal convection has been extensively studied quantitatively for almost a century [2]. What does through flow add to this problem? If the Reynolds number is sufficiently small, the rolls maintain
their form but are transported down the cell [3]. It is this advection which gives rise to the concepts of convective and absolute instabilities which are generic in open flow systems. They are best illustrated by a simple thought experiment. The underlying mathematical concepts will be made more precise later.

There are three parameter regimes in the Rayleigh-Reynolds parameter space of experimental interest. They are shown schematically in Figure 1-2. Assume that the system is initially in the conduction state, so that the

![Diagram of flow and temperature profiles](image)

**FIGURE 1-2. Schematic of Absolute and Convective Instabilities**

Examples of stable, convectively, and absolutely unstable flow. In each example the time evolution of a pulse created at time $t_1$ is considered.

1. For large enough Reynolds numbers, there are shear instabilities which are not considered here. This is known as the Orr-Sommerfeld problem.
only fluid motion is that generated by the applied pressure gradient. Consider the time evolution of a small, localized temperature or velocity perturbation. This perturbation could take the form of a weak pair of convection rolls, but the exact form of the perturbation is not relevant. If the temperature difference is below some critical value, $\Delta T < \Delta T_{\text{conv}}$, the amplitude of the perturbation will decay in addition to being advected downstream by the applied flow. In this case the system is said to be stable. Next consider what happens if the temperature difference is above the critical temperature difference $\Delta T > \Delta T_{\text{conv}}$. The perturbation will now grow in the co-moving frame. However if the system is only slightly above $\Delta T_{\text{conv}}$, the pulse will be transported faster than it grows and spreads. Hence, the growing pulse will always traverse and exit the cell. Even though the pulse is growing in the co-moving frame, the amplitude will eventually decay at every point in the (spatially finite) lab frame. This is called a *convectively unstable* state. The choice of terminology is unfortunate since this definition should not be confused with convective motion. Convective instabilities are generic to open flow systems, even those which have nothing to do with thermal convection. The last state occurs when the temperature difference is above another critical value, $\Delta T > \Delta T_{\text{abs}} > \Delta T_{\text{conv}}$, where the expansion of the pulse due to its growth is rapid enough to outpace the advection. This state is an *absolutely unstable* state. In the lab frame, the perturbation grows so rapidly
that it backs up against the flow. An important observation is that a convectionally unstable system is exquisitely sensitive to noise. In the absence of noise the cell will always empty out. Conversely, if there is source of sustained perturbations, they will always grow to saturation at some distance downstream. The observed pattern is then entirely noise sustained [4]. The system is then acting as a deterministic amplifier of a purely stochastic driving force. Therefore if the cell is long enough that one can observe the growth of the perturbations to saturation, one should always expect to find noise sustained structure. Furthermore, even in the absence of external noise from the laboratory, the system will amplify Brownian motion. A central goal of this experiment is to measure the strength of the noise driving the observed pattern and compare it to thermal noise. The larger the noise source, the more rapidly the amplitude of the pattern will grow to saturation. One may use the onset length as a measurement of the strength of the noise. It is analogous to the idea that if one balances a pencil on its point, the time that it takes to fall depends on the accuracy with which it was originally balanced and the noise which acts to destabilize it. The role of the applied flow is to couple space and time so that if one observes the convection near the inlet, it is analogous to examining the balanced pencil a short time after it is released. Similarly, observing at the system near the outlet is analogous to examining the pencil a long time after it is released.
1.1.3 Other Systems which Show Convective and Absolute Instabilities

Convective and absolute instabilities are not limited to thermal convection with through-flow. They are a generic behavior of perturbation growth in open flow systems [5]. For example, the vortices which grow on a jet of fluid injected into quiescent medium also show convective instabilities [6]. The vortices are undetectably small at the inlet of the jet and grow to saturation as they are transported downstream. In this classic experiment, however, the growth rate of the vortices depends on the distance from the nozzle. The system might be stable near the inlet, convectively unstable somewhere downstream, and absolutely unstable further downstream. This makes the analysis of this system much more difficult than the problem of thermal convection with through flow where the growth rate of the convection rolls depends only on the applied Rayleigh number, which is constant along the length of the channel. For the convection problem, whether the system is convectively or absolutely unstable is a global property of the system, instead of a local property of the flow.

Another well studied example is that of Taylor vortex flow (TVF) with axial flow [7, 8]. In this experiment the fluid is constrained between two concentric cylinders. The inner cylinder is rotated and a pressure gradient is applied along the axis of the cylinders causing a mean axial flow. At a critical rotation rate, toroidal Taylor vortices are generated along the axis.
As with convection with through flow, the vortices are undetectably small near the inlet and grow to saturation as they drift downstream. This system is particularly closely related to the case of thermal convection. It has been shown that both systems, near onset, are well described by a complex Ginzburg-Landau equation which describes the evolution of the amplitude of roll structures [9, 10]. Moreover, as in convection with through flow, perturbations grow with a constant spatial growth rate as a function of distance (until near saturation). Although this systems has been studied with much success, the noise strength driving the rolls has been found to be approximately 270 times larger than thermal noise [11]. This may be due to the somewhat complicated boundary conditions at the inlet. Since the end caps of the cylinder do not rotate, there is an Eckmann vortex generated there. The injected fluid must “spin-up” to match the rotation of the fluid in the body of the apparatus. I will show in Chapter 2 that the boundary conditions for the convection problem are better defined.

Yet another system which shows a convective instability is convection in binary mixtures. This system is different from the previously discussed systems in that the roll motion is not generated by an externally applied pressure gradient. Rather, in a range of fluid parameters, the conduction state becomes unstable via a Hopf bifurcation to a state of travelling rolls. The Hopf bifurcation comes about because there is a large difference
between the thermal diffusion coefficient which sets the time scale for temperature fluctuations to dissipate and the mass diffusion coefficient which sets the time scale for concentration fluctuations to dissipate. The temperature and concentration field can get out of phase with each other giving rise to the travelling rolls. One difference between this system and convection with through flow is that the travelling wave frequency decreases with increasing amplitude. This means that if one lets the convection rolls grow to saturation, the small amplitude linear waves will collide with the large amplitude saturated waves complicating the analysis. For this reason, the experiments of Schöpf and Rehberg, which were designed to look for noise driven rolls, were performed in a very short cell (length = 6 depth) which is bounded by spatial ramps [12]. They claim to have found a noise strength which is consistent with thermal noise, however this finding is somewhat controversial as their experiments are dominated by end effects. The analysis requires a quantitative understanding of the effect of reflection from the ramps. Moreover, the cells are very narrow (width = 1/2 depth) in the transverse direction and the theoretical predictions for the noise strength do not consider the effect of the horizontal boundaries [13]. Lastly, there is the interaction of right and left travelling waves. This subtlety is not present in the problem of convection with through flow where the rolls only travel in the direction of the applied flow.
1.2 Linear Theory

1.2.1 The Boussinesq Approximation

In this section the dynamical equations of convection in an incompressible fluid are derived and an outline of their solution is presented. This is not intended to be an in depth exposition of the theory, but should serve to motivate the subsequent discussion of the nonlinear theory. The fluid must obey the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1-3)
\]

which expresses conservation of mass, and the Navier-Stokes equation in the presence of a gravitational field

\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \rho \mathbf{v} \nabla^2 \mathbf{v} - \nabla P - \rho g \hat{z}, \quad (1-4)
\]

which expresses the conservation of momentum. It is really a simple statement of Newton's third law, \( \mathbf{F} = m \mathbf{a} \). The right side of Eq 1-4 is the force per volume on a small fluid element, and the left hand side is the product of density and the acceleration of the element. The last expressions give the transport of heat where we assume that velocities are small enough that viscous heating, which is proportional to the square of the velocity, may be neglected.
\[ j = -\lambda \nabla T \]
\[ \rho C_p \left( \frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T \right) = \nabla \cdot j. \] (1-5)

These can be combined as

\[ \frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T = -\frac{1}{\rho C_p} \nabla \cdot (\lambda \nabla T), \] (1-6)

which expresses the conservation of the internal energy of the fluid. These equations may be considerably simplified by employing an observation, original due to Boussinesq [14]. He observed that the density is typically a weak function of temperature so that to linear order one may write an expansion

\[ \rho = \rho_0 [1 - \alpha (T - T_0)], \] (1-7)

where \( \rho_0 \) is the density at some reference temperature \( T_0 \). For the current problem, the reference temperature will be chosen as the temperature at the midplane of the cell. The thermal expansion coefficient \( \alpha \) is very small. For water at 25 °C it is \( 2.58 \times 10^{-4} \, \text{K}^{-1} \). If the temperature difference between the plates in the convection cell is only a few degrees, then to a very good approximation, the expansion may be ignored in the continuity equation, and the fluid may be considered to be incompressible.

However in the momentum equation it is scaled by the acceleration due to gravity, where it provides the buoyancy force, and it can not be ignored. The Boussinesq approximation, therefore assumes that all of the
material parameters are independent of temperature, except for the density in the momentum equation, where it provides the driving force for the convection.

This approximation is unfortunately not self consistent, but it yields a considerable simplification. In particular, the continuity equation now states that the divergence of the velocity vanishes

$$\nabla \cdot \mathbf{v} = 0.$$  \hspace{1cm} (1-8)

The Navier-Stokes equation can be rewritten in terms of the difference between the pressure and the hydrostatic pressure. Write

$$P = - \rho g z + p,$$  \hspace{1cm} (1-9)

so that the Navier-Stokes equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{\nabla P}{\rho_0} + \nu \nabla^2 \mathbf{v} + \alpha g(T - T_0) \hat{z}.$$  \hspace{1cm} (1-10)

The last term is the buoyant force. The heat transport equation is also simplified to

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T = - \kappa \nabla^2 T,$$  \hspace{1cm} (1-11)

where $\kappa = \lambda / \rho_0 C_p$ is the thermal diffusivity. Equations Eq 1-8, Eq 1-10, and Eq 1-11 comprise the Boussinesq approximation. They may be further simplified by the introduction of non-dimensionalized variables as was originally done by Rayleigh [1]. There are several dimensionless
quantities which can be constructed from the fluid parameters and the dimensions of the cell. Change variables to

\[ x' = x/d \]
\[ t' = t \left[ \frac{d^2}{\kappa} \right] \]
\[ v' = v/[(\kappa/d)] \]
\[ T' = T \left[ \frac{\kappa \nu}{\alpha g d^3} \right] \]
\[ P' = P \left[ \rho_0 \left( \frac{\kappa}{d} \right)^2 \right], \]

(1-12)

where all of the primed variables are dimensionless. The Boussinesq equations become

\[ \nabla' \cdot v' = 0 \]
\[ \frac{\partial v'}{\partial t'} + (v' \cdot \nabla')v' = -\nabla' p' + Pr \nabla'^2 v' - Pr (T' - T_0') \Delta \]
\[ \frac{\partial T'}{\partial t'} + (v' \cdot \nabla')T' = -\nabla'^2 T', \]

(1-13)

where Pr = \nu/\kappa is known as the Prandtl number and is only a function of the fluid properties. It measures the ratio of the momentum diffusion rate to the thermal diffusion rate. For water, it ranges between about 2 and 10 depending on the temperature. For materials with low viscosity and high thermal conductivity such as liquid helium, it is 0.6. For mercury it is 0.02. For poorly conducting, viscous fluids, such as silicone oil, it can be as large as several hundred.
To simplify the notation, I will drop the primes, but henceforth all of the physical fields will be understood to be the dimensionless quantities. When I need to refer to the dimensional fields I will apply the superscript "ph" to make it explicit which quantities are physical and which are dimensionless. It is important to note that the only nonlinear terms in these equations are the \((v \cdot \nabla)v\) and the \((v \cdot \nabla)T\) terms. One can simplify the solution to the Boussinesq equations by separately solving for the linear parts of the equations and then writing a new set of nonlinear equations for the deviations from the linear solutions. Make this idea concrete by writing the velocity, temperature, and pressure fields as

\[
\begin{align*}
\mathbf{v} &= \mathbf{v}_l + \mathbf{v} \\
T &= T_l + \theta \\
p &= p_l + p.
\end{align*}
\] (1-14)

The \(l\) subscripted variables are the solutions to the Boussinesq equations with the nonlinear terms removed. One may now write dynamical equations for the fields \(v, \theta,\) and \(p\) as

\[
\begin{align*}
\nabla \cdot v &= 0 \\
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}_l + (\mathbf{v}_l \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= -\nabla p + Pr \nabla^2 \mathbf{v} - Pr \theta^2 \\
\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla)T_l + (\mathbf{v}_l \cdot \nabla)\theta + (\mathbf{v} \cdot \nabla)\theta &= -\nabla^2 \theta.
\end{align*}
\] (1-15)

1.2.2 Calculation of the Baseflow

In this section I will calculate the flow associated with the time independent solution to the linear parts of the Boussinesq equations. This solu-
tion is traditionally known as the baseflow. The baseflow must satisfy Eq 1-8, Eq 1-10, and Eq 1-11 with the nonlinear terms and the time dependent terms eliminated

\[ \nabla \cdot \mathbf{v}_I = 0 \]
\[ 0 = - \nabla p_I + \text{Pr} \nabla^2 \mathbf{v}_I + \text{Pr} (T_I - T_0) \mathbf{x} \]
\[ 0 = \nabla^2 T_I. \]  

The \( I \) subscript refers to the fact these quantities are solutions to the linear equations. The boundary conditions are

\[ \mathbf{v}_I(y = \pm \beta/2) = \mathbf{v}_I(z = \pm 1/2) = 0 \]
\[ T_I(z = 1/2) = T_t \]
\[ T_I(z = -1/2) = T_b, \]
\[ \mathbf{x} \cdot \nabla p_I = -p_x. \]  

where \( T_t \) and \( T_b \) are the nondimensionalized temperatures of the top and bottom plates and, \( p_x \) is the nondimensionalized pressure gradient in the \( x \) direction which drives the through flow from \( x = -\infty \) to \( x = +\infty \). The solution to the thermal equation in Eq 1-16 is

\[ T_I = \frac{T_b + T_t}{2} + (T_b - T_t)z. \]  

The quantity \((T_b + T_t)/2\) is simply the dimensionless midplane temperature \( T_0 \). The dimensionless quantity \( T_b - T_t \) is the Rayleigh number as may be seen by using Eq 1-12 to write it in terms of the physical quantities as
\[ \text{Ra} = T_b - T_f = \frac{\alpha \gamma d^3 A T^\text{ph}}{\kappa V}, \quad (1-19) \]

where I have used the notation \( A T^\text{ph} \) as a reminder that this is the physical temperature difference across the cell (measured in °C). In terms of the Rayleigh number Eq 1-18 is simply

\[ T_f = T_0 + \text{Ra} z. \quad (1-20) \]

The solution to the velocity equation in Eq 1-16 is particularly simple for the case of the infinitely wide channel, \( \beta = \infty \). Assume that the pressure can be written as the sum

\[ p_l(x) = P_x(x) + P_z(z). \quad (1-21) \]

Integrating the pressure boundary condition in the \( x \) direction gives

\[ P_x(x) = -xp_x. \quad (1-22) \]

Since the applied pressure gradient is in the \( x \) direction and only the solution with no convection is being considered, look for a velocity in the \(-x\) direction. Furthermore, by symmetry, the velocity can only be a function of \( z \). Therefore one has the form \( v_l(x) = \nu(z)x \). With this assumption one may write the middle equation of Eq 1-16 in components as

\[ \nu''(z) = -\frac{p_x}{Pr} \quad (1-23) \]

\[ p'(z) = \text{Pr Ra} \frac{z}{2}. \]

These may be integrated to give the velocity and pressure as
\[ v(z) = -\frac{p_x}{Pr} \left( z^2 - \frac{1}{4} \right) \]  
\[ p(z) = Pr Ra z^2. \]  
(1-24)

This results is more conveniently expressed in terms of the mean velocity in the cell, rather than the applied pressure gradient since the flow rate is what is measured in the experiment. Integrating Eq 1-24 over the cell volume gives

\[ \bar{v} = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(z) \, dz = \frac{p_x}{6Pr}. \]  
(1-25)

Therefore one may write Eq 1-24 in terms of the mean velocity as

\[ v = 6 \bar{v} \left( \frac{1}{4} - z^2 \right) \hat{x}. \]  
(1-26)

It is interesting to note that the maximum velocity is attained in the center of the channel and \( v_{\text{max}} = \frac{3}{2} \bar{v} \). One may express the nondimensional velocity (which is sometimes known as the Péclet number \( \bar{v} = Pe = \frac{v_{\text{th}} d}{\kappa} \)) in terms of the Reynolds number and the Prandtl number by using the expressions in Eq 1-12. One has

\[ \bar{v} = Re Pr \cdot \]  
(1-27)

The linear solution may now be summarized as

\[ v_l = 6 Re Pr \left( \frac{1}{4} - z^2 \right) \hat{x} \]  
\[ T_l = T_0 + Ra z \]  
\[ p_l = 6Re Pr^2 x + Ra Pr z^2. \]  
(1-28)
The generalized expression for the finite aspect ratio is substantially more complicated. See APPENDIX A Calculation of Baseflow for a Finite Width Channel on page 251. Therefore, the present discussion will be restricted to the infinite aspect ratio case. The solution Eq 1-28 can be inserted in Eq 1-15 to get an equation for the nonlinear system. One has

\[
\begin{align*}
\nabla \cdot \mathbf{v} &= 0 \\
\frac{\partial \mathbf{v}}{\partial t} - 12 \text{Re } Pr z v_z \mathbf{x} + 6 \text{ Re } Pr \left( \frac{1}{4} - z^2 \right) \frac{\partial \mathbf{v}}{\partial x} + (\mathbf{v} \cdot \nabla) \mathbf{v} \\
&= -\nabla p + Pr \nabla^2 \mathbf{v} - \text{Pr } \theta \mathbf{e} \\
\frac{\partial \theta}{\partial t} + Ra v_z + 6 \text{ Re } Pr \left( \frac{1}{4} - z^2 \right) \frac{\partial \theta}{\partial x} + (\mathbf{v} \cdot \nabla) \theta &= -\nabla^2 \theta.
\end{align*}
\tag{1-29}
\]

One may now compute when this system becomes unstable by performing a linear stability analysis. Consider the time evolution of small sinusoidal perturbations of the form

\[
\begin{pmatrix}
\mathbf{v}(\mathbf{x}) \\
\theta(\mathbf{x}) \\
\rho(\mathbf{x})
\end{pmatrix} =
\begin{pmatrix}
\Phi(z) \\
\hat{\theta}(z) \\
\hat{\rho}(z)
\end{pmatrix} e^{i(k \cdot \mathbf{x} + \sigma t)},
\tag{1-30}
\]

where the wavevector \( \mathbf{k} \) is in the plane \( \mathbf{k} = (k_x, k_y, 0) \), and \( \sigma \) is the growth rate of the perturbation. The system becomes unstable when \( \sigma \) becomes positive. The first wavevector which achieves a positive growth rate is known as the critical wavevector. Insert Eq 1-30 into Eq 1-29 and linearize with respect to the "hatted" variables. The PDE's become coupled ODE's.
\( \mathbf{k} \cdot \mathbf{v} = 0 \)

\[
\begin{align*}
\sigma \dot{v} - 12 \text{Re Pr} z \dot{v}_z & + 6 \text{Re Pr} \left( \frac{1}{4} - z^2 \right) i k_x \dot{v} \\
& = -\nabla \phi + \text{Pr}(- \dot{v}_x k_x^2 \dot{v} - \dot{v}_y k_y^2 \dot{v} + \dot{v}_z' \dot{z}) - \text{Pr} \dot{z} \dot{z} \\
\sigma \ddot{\theta} + \text{Ra} \dot{v}_z + 6i \text{Re Pr} \left( \frac{1}{4} - z^2 \right) k_x \ddot{\theta} &= -k^2 \ddot{\theta} - \ddot{\theta}''.
\end{align*}
\]  (1-31)

The critical Rayleigh number and wavevector, as a function of the applied Reynolds number, are given by the \( \theta \) and the \( z \) component of the velocity equation which when written out in components are

\[
\begin{align*}
\sigma \dot{v}_z + 6i \text{Re Pr} \left( \frac{1}{4} - z^2 \right) k_x \dot{v}_z &= -\frac{d \Phi}{dz} + \text{Pr} \dot{v}_z'' - \text{Pr} \dot{\theta} \\
\sigma \ddot{\theta} + \text{Ra} \dot{v}_z + 6i \text{Re Pr} \left( \frac{1}{4} - z^2 \right) k_x \ddot{\theta} &= -k^2 \ddot{\theta} - \ddot{\theta}''.
\end{align*}
\]  (1-32)

These ODE's were numerically solved by Luijx [15] for nonzero Reynolds number and analytically by Schlüter, Lortz, and Busse for the simple Bénard problem [16]. See APPENDIX B Eigenfunctions of the Simple Bénard Problem on page 257. Without solving Eq 1-32, a couple of simple observations can be made. For the case of longitudinal rolls, \( k_x = 0 \) and the Reynolds number drops out of Eq 1-32. The equations reduce to the case of simple Rayleigh-Bénard convection. Therefore the critical temperature difference and critical wavevector are independent of the applied flow. For the transverse rolls \( k_x \neq 0 \). The system has the symmetry of being independent of the direction of the flow, \( \text{Re} \rightarrow -\text{Re} \) so the solutions must be functions of \( \text{Re}^2 \). Furthermore, the critical Rayleigh
number must agree with the simple Bénard problem as the flow is set to zero. Therefore, near onset, one must have

\[
\begin{align*}
Ra_c^{TR} &= Ra_c + a Re^2 + O(Re^4) \\
Ra_c^{LR} &= Ra_c,
\end{align*}
\]

(1-33)

where \(Ra_c^{TR}\) is the critical Rayleigh number of the transverse rolls, \(Ra_c^{LR}\) is the critical Rayleigh number of the longitudinal rolls, and \(Ra_c = 1708\) is the critical Rayleigh number of the Bénard problem. The coefficient \(a\), was numerically computed by Luijx for a range of Prandtl numbers. The implication is that in the absence of the lateral sidewalls, the critical Rayleigh number of the longitudinal rolls is always lower than for transverse rolls. Therefore in a laterally infinite system, such as the geophysical problem of cloud streets which result from convection in the atmosphere, one would expect the rolls to align with the prevailing wind. Luijx also computed the effect of the sidewalls and found that the walls stabilize the conduction state so that \(Ra_c\) is increased. However the stabilization is larger for the longitudinal rolls, than for the transverse rolls. Conceptually this is easy to understand. The longitudinal rolls have a velocity maximum near the sidewalls which creates a region of high shear. The resulting viscous stress damps the convective motion and allows more time for heat to conduct through the fluid, reducing its buoyancy. In contrast the transverse rolls have a node in the center of the rolls near the wall. The mean viscous stress is therefore lower than for the longitudinal rolls. The situation is
shown schematically in Figure 1-3 which shows the critical Rayleigh

![Graph showing critical Rayleigh number as a function of Reynolds number for infinite and finite width channels.](image)

**FIGURE 1-3. Critical Rayleigh Number as Function of Reynolds Number for Infinite and Finite Width Channels**

For the infinite width channel, the critical Rayleigh number for longitudinal rolls is always lower than for the transverse rolls. Therefore the rolls will always align with the prevailing "wind". When the cell has a finite width, both stability boundaries are raised, but the longitudinal roll boundary is raised more than the transverse roll boundary. This leads to a co-dimension two point where they cross.

dnumber as a function of the applied Reynolds number for infinite and finite width channels. For infinite aspect ratio case, both the longitudinal and the transverse roll boundaries meet at $Re = 0$ where the problem reduces to the simple Bénard problem. When one adds sidewalls, both boundaries are raised, leading to a co-dimension two point where they cross.
1.3 Weakly Nonlinear Theory

1.3.1 Amplitude Equations

In 1969, Newell, Whitehead and Segel showed that one can compute the time evolution, growth and saturation of the convection rolls in the Rayleigh-Bénard problem (without applied flow) near the onset by performing a weakly nonlinear analysis [17, 18]. They wrote the dynamical fields in the form of a product of a slowly varying complex amplitude $A(x, t)$ times the lowest order linear solution

$$\begin{pmatrix} v(x) \\ \theta(x) \\ p(x) \end{pmatrix} = A(x, t) \begin{pmatrix} \hat{v}(z) \\ \hat{\theta}(z) \\ \hat{p}(z) \end{pmatrix} e^{ik \cdot x}. \tag{1-34}$$

They then showed that the resulting dynamical equations for the amplitude take the form of a Ginzburg-Landau equation. If one chooses the $x$ direction parallel to $k$, and $y$ along the roll axis one has

$$\tau_0 \frac{\partial A}{\partial t} = eA + \xi_0^2 \left( \frac{\partial}{\partial x} - \frac{i}{2q_c} \frac{\partial^2}{\partial y^2} \right)^2 A - g|A|^2 A. \tag{1-35}$$

where the coefficients were computed for free-free boundary conditions.

Then in 1980, Cross rederived this equation and its coefficients for rigid-rigid boundary conditions which is more applicable to experiments in confined geometries [19].

23
There is a great conceptual simplification from this formalism. Since the amplitude is complex, it can be written as two real fields $a$ and $\varphi$ as $A = ae^{i\varphi}$. The magnitude $a$ measures the strength of the convection and the phase $\varphi$ measures the deviation of the system from the critical wavevector of the underlying roll structure. One has, in effect, coarse grained the system over the behavior of many rolls much in the same way the hydrodynamics coarse grains the behavior of individual particles into the dynamics of a continuous field. In hydrodynamics, the exact nature of the interparticle forces becomes irrelevant; they only contribute to the phenomenological coefficients of the fluid dynamical equations such as the viscosity, density, and thermal conductivity. In a similar way, the exact nature of the underlying roll structure, or even the physics which gives rise to rolls, becomes unimportant, contributing only to the numerical values of the coefficients of the amplitude equation. This simplification means that one can apply a consistent formulation and analysis for many pattern forming systems, if their dynamics can be reduced to an amplitude equation.

1.3.2 Results of Müller et. al.

The amplitude equation formalism was extended to the case of convection with through flow by Müller, Lücke, and Kamps in 1989 [20]. They considered the amplitude of the transverse rolls which occur for low flow
rates. Their numerical results were computed for Prandtl number of 1.0. This work was later extended by Müller, Tveitereid, and Trainoff to include the dynamics which couple the longitudinal and transverse rolls near the co-dimension two point where both modes have a finite amplitude [21]. The results of Müller et. al. will be used extensively in this work so they are reproduced here for convenience. They computed a coupled amplitude equation and found

\[
\phi_1 = A\hat{\phi}_A(z)e^{i(kx-\omega t)} + B\hat{\phi}_B(z)e^{iy} + \text{c.c.},
\]

(1-36)

where \(A\) and \(B\) are the amplitudes of the transverse and longitudinal rolls. The eigenfunctions \(\hat{\phi}_A(z)\) and \(\hat{\phi}_B(z)\) are solutions of the linearized equations Eq 1-32. The equations for the amplitudes are given by the coupled CGLs

\[
\begin{align*}
\tau_A \left( \frac{\partial}{\partial t} + \nu_A \frac{\partial}{\partial x} \right) A &= \left( \epsilon - \epsilon_c^T \right)(1 + i\nu_0)A + \xi_A^2(1 + i\nu_1) \frac{\partial^2 A}{\partial x^2} \\
&\quad - (1 + i\nu_2)|A|^2 A - \beta_A(1 + i\nu_3)|B|^2 A \\
\tau_B \left( \frac{\partial}{\partial t} + \nu_B \frac{\partial}{\partial x} \right) B &= \left( \epsilon - \epsilon_c^L \right)B + \xi_B^2 \frac{\partial^2 B}{\partial x^2} + \chi_B \frac{\partial^3 B}{\partial x^3} - \lambda_B \frac{\partial^4 B}{\partial x^4} \\
&\quad - |B|^2 B - \beta_B |A|^2 B.
\end{align*}
\]

(1-37)

The coefficients have been computed numerically for the Prandtl number relevant to this experiment, \(\text{Pr} = 5.8\)
\[
\begin{align*}
\tau_A &= 0.0554 + 7 \times 10^{-5} \text{Re}^2 & c_0 &= 0.187 \text{Re} \\
v_A &= 7.507 \text{Re} & c_1 &= 0.0382 \text{Re} \\
\xi_A^2 &= 0.1482 + 1.16 \times 10^{-3} \text{Re}^2 & c_2 &= -0.0030 \text{Re} \\
\beta_A &= 1.2486 + 2.5 \times 10^{-3} \text{Re}^2 & c_3 &= -0.0450 \text{Re} \\
\tau_B &= 0.0554 - 2.7 \times 10^{-8} \text{Re}^2 & v_B &= 7.435 \text{Re} \\
\xi_B^2 &= 1.0 \times 10^{-3} \text{Re}^2 & \xi_B &= 2.1 \times 10^{-4} \text{Re} \\
\beta_B &= 1.2486 + 2.3 \times 10^{-3} \text{Re}^2 & \lambda_B &= 3.8 + 10^{-5} \text{Re}^2.
\end{align*}
\] (1-38)

The experiment was performed for Reynolds numbers below the co-dimension two point, where the longitudinal rolls are strongly damped. Therefore, for the purposes of comparing the theory to experiment, the longitudinal roll amplitude will be set to zero and only the first equation in Eq 1-37 will be used. The resulting amplitude equation is identical to that in Ref [20] but the coefficients have been computed for the experimental value of the Prandtl number.

1.4 Stochastic Forcing of the Amplitude Equation

It was originally pointed out by Landau and Lifschitz that the Navier-Stokes equation is an incomplete description of fluid dynamics [22, 23]. In particular, the coupling of thermal fluctuations to the hydrodynamic fields must be included. Landau and Lifschitz showed that one must add a stochastic forcing term to both the stress tensor and the heat current. The stress tensor and heat flux are replaced by (using the notation of Ref. [22])
\[ \sigma'_{ik} \rightarrow \sigma'_{ik} + s_{ik} \]
\[ q_i \rightarrow q_i + g_i, \]

where the generalized stress tensor and heat flux are given by

\[ \sigma'_{ik} = \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_i} \right) + \zeta \frac{\partial v_l}{\partial x_i} \delta_{ik} \]
\[ q = -\kappa \nabla T, \]

and \( s_{ik} \) and \( g_i \) are random quantities which are thermally driven. The coefficients \( \eta, \zeta, \) and \( \kappa \) are the dynamic viscosity, the bulk viscosity, and the thermal diffusivity, respectively. The stochastic terms obey the correlation functions

\[ s_{ik}(r_1, t_1) s_{lm}(r_2, t_2) = 0 \]
\[ g_i(r_1, t_1) g_k(r_2, t_2) = 2\kappa T^2 \delta_{ik} \delta(t_1 - t_2) \delta(r_1 - r_2) \]
\[ s_{ik}(r_1, t_1) s_{lm}(r_2, t_2) = 2T \left[ \eta \left( \delta_{il} \delta_{km} + \delta_{im} \delta_{kl} \right) + (\zeta - 2\eta/3) \delta_{ik} \delta_{lm} \right] \]
\[ \times \delta(r_1 - r_2) \delta(t_1 - t_2). \]

The bars over the quantities on the left hand side denote averaging in the statistical sense. One averages over all of the values that the random variables may assume, weighted by their probabilities. These expressions are completely general allowing for arbitrary geometries as well as for compressible flow. In these expressions the temperature has been measured in units of Boltzmann's constant \( k_B \). For incompressible fluids, the stress tensor reduces to

\[ \sigma'_{ik} = -p \delta_{ik} + \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right). \]
This formalism is a perfectly general description of the fluctuations of a simple fluid. At the same time that Landau and Lifshitz published the preceding results, a similar derivation of the stochastic terms was performed by Khaltatnikov who was interested in the interaction of the hydrodynamic fluctuations with superfluid helium [24]. He obtained a generalization of the Landau and Lifschitz results which include the thermal fluctuations of the composition of the fluid. His formalism is applicable to the case of convection in binary fluid mixtures [12].

The problem of how to use the Landau and Lifshitz theory to compute the effect of the stochastic terms on the amplitude equations was explored by Zaitsev and Shliomis in 1971 [25] and Graham in 1974 [26]. Graham considered the case of free slip boundary conditions on top and bottom and an infinite lateral extent. He showed that the amplitude equation acquires an additive white noise term so that Eq 1-35 becomes

\[ \frac{\partial A}{\partial t} = \varepsilon A + \xi_0^2 \left( \frac{\partial}{\partial x} - \frac{i}{2q_c} \frac{\partial^2}{\partial y^2} \right)^2 A - g|A|^2 A + f_A. \]  

(1-43)

The term \( f_A \) is a complex Gaussian white noise random variable satisfying the correlations

\[ \langle f_A(x, y, t)f_A^*(x', y', t') \rangle = 2F_A \xi_0^2 \tau_0 \delta(x - x') \delta(y - y') \delta(t - t') \]

\[ \langle f_A(x, y, t)f_A(x', y', t') \rangle = \langle f_A(x, y, t) \rangle = 0 \]  

(1-44)

\[ F_A = F_{th}/2q_c \xi_0. \]
This work was then extended by van Beijeren and Cohen to the physically relevant case of rigid boundaries [27]. Their expression for the stochastic noise term can be written in the form

\[ F_{th} = \frac{k_B T}{\rho d v^2} \frac{2Prq_c}{\xi_0 \tau_0 Ra_c}. \]  

(1-45)

The prefactor in brackets can be understood on dimensional grounds as the ratio of two energy scales. The first is the energy scale of the thermal fluctuations \( k_B T \), and the second is an energy scale associated with viscous damping \( (\rho d^3)(v/d)^2 \). The second factor in Eq 1-45 represents the coupling of the noise to the amplitude equation. It is written in terms of the amplitude equation coefficients, the Prandtl number and the critical Rayleigh number. The only effect of the top and bottom boundary conditions is to change the numerical values of the amplitude equation coefficients [28]. However this expression is correct only for geometries which are unbounded in the plane. For the pseudo-one dimensional geometry in which the experiment was performed the noise strength is modified. Treiber has shown in principle, how to compute the noise strength for a finite width channel, but has only computed an explicit expression for the related problem of Taylor vortex flow with axial through flow [29].

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1. This form is equivalent to that computed by van Beijeren and Cohen even though their form was expressed somewhat differently. Hohenberg and Swift pointed out the remarkable fact that when the noise is written in this way, the expression remains unaltered for both the free and rigid boundary conditions.
1.5 Historical Context

This section will present an abbreviated history of the problem of convection in the presence of thorough flow. It is not intended to be complete, but is only to give the reader a sense of context for this work.

Consider first the case where there is no applied flow and the cell has no lateral boundaries. This is known as the Bénard problem after Henri Bénard who made the first quantitative observations of the onset of convection in this geometry in 1900 [2] although the phenomena had been studied earlier by many others. The first theoretical treatment of the problem was performed by Rayleigh in 1916 [1]. He found that the conduction state becomes unstable to a state of rolls at a finite temperature difference. He used free slip boundary conditions on the top and bottom to show that the critical Rayleigh number where convection begins is given by \( \text{Ra}_c = \frac{27}{4} \pi^4 = 657.5 \) and that, at onset, the convection rolls have a dimensionless wavevector of \( k_c = \pi/\sqrt{2} = 2.221 \). The first calculation of the onset in a cell with rigid top and bottom boundaries was first performed by Low in 1929 [30]. He found that the critical Rayleigh number is given by the solution to a transcendental equation and can only be evaluated numerically. For the rigid-rigid case it is \( \text{Ra}_c = 1707.8 \) and the critical wavevector is \( k_c = 3.117 \).
The case of convection in a finite width channel was considered by Frick and Clever in 1980. The addition of the horizontal boundaries breaks the rotation symmetry in the plane which exists for the infinite system. The rolls aligned with the channel (longitudinal rolls) now have a different critical Rayleigh number than rolls which are perpendicular to the channel (transverse rolls)[31]. They only considered transverse rolls (TR) and numerically computed the critical Rayleigh number as a function of the aspect ratio (width/depth), but the overall trend is that the critical Rayleigh number for TR is a monotonically decreasing function of the width. In 1983, Luijx extended these results to the case of longitudinal rolls (LR) and found that there is a series of local minima in the critical Rayleigh number which correspond to an integral number of rolls fitting into the channel [15]. Luijx also computed the critical Rayleigh number as a function of the applied flow. He was also able to compute the critical wavevector at onset as well as the shape of the three dimensional roll structures. Owing to the complexity of the baseflow (see Eq A-14 on page 254), he only performed the linear stability analysis. The first derivation of the amplitude equation for convection with through flow was performed by Müller, Luecke, and Kamps [20].
1.6 Convective and Absolute Instabilities in the Context of the Amplitude Equation

In Section 1.1.2 I discussed the concepts of *convective* and *absolute* instabilities. These ideas can be made mathematically precise by considering solutions of the Ginzburg-Landau equation. This approach was first discussed by Deissler who was interested in noise sustained structures [4]. Consider the Green function propagator of the linearized CGL in Eq 1-37 for pure transverse rolls \((B = 0)\)

\[
\tau_A \left( \frac{\partial}{\partial t} + v_A \frac{\partial}{\partial x} \right) A = (\epsilon - \epsilon_c^T) (1 + ic_0) A + \xi_A^2 (1 + ic_1) \frac{\partial^2 A}{\partial x^2} + \delta(x) \delta(t).
\]

One has the solution

\[
A(x, t) = \sqrt{\frac{(1 + ic_1) \tau_A}{4 \xi_A^2 \pi t}} \times \exp \left[ (\epsilon - \epsilon_c^T) (1 + ic_0) \frac{t}{\tau_A} - \frac{(x - v_A t)^2 \tau_A}{4 \xi_A^2 (1 + ic_1) t} \right].
\]

Consider the behavior of Eq 1-47. If one rewrites the exponential in the form of a temporal growth and a dispersion \(\exp[\sigma t + ist]\), where \(\sigma\) and \(s\) are both real, one can identify the temporal growth of the pattern at any point, \(x\) as
\[ \sigma(x, t) = \frac{1}{t} \text{Re} \left[ (\varepsilon - \varepsilon_c^T)(1 + ic_0) \frac{t}{v_A} - \frac{(x - v_A t)^2 \tau_A}{4 \xi_A^2 (1 + ic_1 t)} \right] \tag{1-48} \]

\[ \frac{\varepsilon - \varepsilon_c^T}{\tau_A} - \frac{(x - v_A t)^2 \tau_A}{4 \xi_A^2 (1 + c_1^2 t^2)} . \]

The growth rate has a maximum when \( x = v_A t \). In this moving reference frame, the amplitude grows when \( \varepsilon - \varepsilon_c^T \) is positive and decays when it is negative. This defines the \textit{convective instability} boundary as

\[ \varepsilon_{\text{conv}} = \varepsilon_c^T . \tag{1-49} \]

To compute the \textit{absolute instability} boundary, one must consider the temporal growth at a point fixed in the lab reference frame as \( t \to \infty \). The growth rate asymptotically becomes

\[ \sigma(x, t \to \infty) \sim \frac{\varepsilon - \varepsilon_c^T}{\tau_A} - \frac{v_A^2 \tau_A}{4 \xi_A^2 (1 + c_1^2)} . \tag{1-50} \]

The \textit{absolute instability} boundary is given by the condition that the growth at any point in the lab frame is neutrally stable \( \sigma(x, t \to \infty) = 0 \), therefore

\[ \varepsilon_{\text{abs}} = \varepsilon_c^T + \frac{v_A^2 \tau_A^2}{4 \xi_A^2 (1 + c_1^2)} . \tag{1-51} \]
CHAPTER 1 Bibliography


2.1 Introduction

This section addresses the issues involved in the design, construction, and operation of the apparatus. The cell portion of the apparatus is shown in Figure 2-1. It consisted of a sample cell, filled with fluid, that was bounded on the top and bottom by flat plates. The top plate was transparent and the bottom plate had a mirrored surface. There was a glass window on the top of the apparatus that gives optical access to the convecting fluid. An optical system which employs the shadowgraph method visualized the flow.

Sandwiched between the plates was a sidewall that constrained the fluid laterally. The sidewall sealed the fluid in the cell with o-rings on its top and bottom surfaces. The assembly was held in position by a clamp ring which compressed the cell and suspended it from the top plate. This
FIGURE 2-1. Sample Cell
This figure shows a side view of the sample cell mounted in its can. This assembly was immersed in the cooling water bath as shown in Figure 2-2.

assembly was mounted in a can which was filled with air. The empty volume in the cell was filled with foam to prevent convection in the air below the bottom plate. Pressed against the bottom plate was an electrical heater which produced the temperature difference across the cell. The whole can was immersed in a circulating water bath. The water in the bath was regulated to a temperature below that of the bottom plate. This determined the top plate temperature. The water removed the heat dissipated by the bottom plate heater. The cooling water entered a flow distributor ring through a slot in the side (not shown in Figure 2-1). The flow distributor had a wide channel which served to equalize the pressure around the ring. The water exited the flow distributor through 32 1/16
inch x 1/16 inch slots which direct the water radially across the face of the top plate. This arrangement was found to have the smallest thermal gradient across the top surface when heat is flowing through the cell. A thermistor dragged across the cell while the apparatus was running showed that when a temperature difference of 3 °C is maintained across the cell, there was less than a 5 mK variation in the top plate temperature. Moreover, the temperature difference across the region of the cell where convection occurs was less than 1 mK.

The cooling water pump system is shown in Figure 2-2. The water was

![Figure 2-2. Bath Cooling Water Pump System](image)

pumped in a loop. It first entered a heat exchanger which served to cool the water to a temperature a few degrees below the operating temperature. The heat exchanger had two separate water paths that were coupled by a
weak thermal link. One water path contained the bath water. The other flowed through a Brinkman Industries Lauda Model K2/R refrigerated circulator [9]. The circulator was capable of maintaining a constant temperature to within 0.1°C of a specified set point. The weak thermal link acted as a low pass filter. It prevented small, but rapid, temperature fluctuations of the circulator water from being transmitted to the bath. The cooling water then passed through a cartridge filter which removed small particles from the flow. It then passed through a heater which was used to regulate the bath temperature.

The temperature controlled fluid entered the cell flow distributor, jetted across the top of the cell sapphire and was collected on the right side of the apparatus where it flowed back to the pump, completing the circuit. The fill tube was connected to the low pressure, inlet side, of the pump and was open to the atmosphere.

The sample fluid flow system (described in Section 2.6.1 on page 91) entered the apparatus through a 3/8 inch diameter polyethylene tube which passed through the cooling water bath. Since the diameter of the tubing was large, the sample fluid flowed very slowly and was in thermal equilibrium with the bath water. It connected to a stainless steel connector which housed a 4 inch long, 0.010 inch diameter, stainless steel capillary which was the dominant flow impedance in the sample flow system.
The temperature of the capillary was regulated to the same precision as the bath fluid, therefore the viscosity of the fluid, and correspondingly the impedance of the capillary, was virtually constant. This impedance, combined with the constant hydrostatic pressure head produced by the flow system, determines the flow rate.

The sample fluid then connected to a 0.070 inch diameter teflon tube which entered the cell through a hole in the center of the bottom plate. The fluid flowed along channels cut from the sidewall until it reached the periphery. It then exited the cell through a hole near the edge. The sidewall channel shape is shown in Figure 2-3. The reason for the rather complicated serpentine shape is discussed in Section 2.3 on page 47.

Thermistors embedded in the bottom plate assembly and in the bath water allow one to precisely measure the temperature difference across the cell. One may also measure the power dissipated in the bottom plate heater to determine the heat current through the cell.

The rest of this chapter will discuss various aspects of the design which make this apparatus unique or were essential to the success of the experiment. I will also discuss some of the methods employed to characterize its performance. I will describe the shape and construction of the side wall, the design of the bottom plate heater, the sample fluid pump system, and the shadowgraph apparatus which visualizes the convecting flows.
2.2 Bath Temperature Regulation System

The cooling water bath was regulated to an rms deviation of 0.5 mK. This performance was achieved by a computer controlled servo loop. The loop consisted of a thermistor which sensed the temperature at the inlet of the apparatus, a computer to compute the required heater power, a heater, and interfacing electronics.

The heater power was computed by an IBM PC AT compatible computer running at 12 MHz. The PC was running the QNX 2.2 operating system from QNX Software [10]. This operating system allows one to perform
true real time multitasking so it is possible to run multiple temperature controllers, data gathering, and analysis tasks simultaneously. The experiment ran two temperature controllers; one for the bottom plate and one for the cooling water bath. I will describe the bath controller since the bottom plate controller used the same algorithm. The computer also took shadowgraph images at precisely defined intervals and performed simple data analysis. For more detailed analysis, the data was transferred to a network of NeXT computers [11].

The PC was equipped with a Scientific Solutions Labmaster board which included, among other devices, two 12 bit digital to analog converters (DAC’s) [12]. It also had an IEEE 488 interface card which was used to communicate with standard laboratory instrumentation [13]. There was also a PCeye® image grabber card from Chorus Data Systems which was used to capture images as a part of the shadowgraph imaging system (See Image Processing on page 107.) [8].

The cooling water servo sensed the water temperature, just after the heater, at the inlet of the apparatus. It was put as close to the heater as possible so that there was only a short time delay between changes in the heater power and when those changes would appear at the thermistor. For this same reason, the thermistor had a relatively small thermal mass so that its temperature would not lag behind the water temperature. On the
other hand, one must insure that the water is well mixed before it comes in contact with the thermistor body, otherwise, the local temperature fluctuates and one gets a poor measurement of the mean temperature of the fluid. Indeed when one operated the servo with more power dissipated in the heater, the stability of the temperature servo decreased. The stability was measured by monitoring the resistance of an additional thermistor placed near the apparatus' outlet.

The temperature sensor was a nominally 100 KΩ thermistor from Penwil Electronics [6]. The resistance of the thermistor could be measured in increments of 0.1 Ω by a Keithley model 196 digital multimeter [14]. However, due to electrical pick up in the cables, the relative precision of the measurement was only about 1.0Ω. This corresponds to a temperature resolution of about 0.2mK. The resistance was sampled every 3 seconds, and the digital value was transferred to the computer using the IEEE-488 bus interface. The PC used the digital value to compute a new heater power using a proportional-integral algorithm. The power at time step $n$, denoted by $P_n$, is given by

$$P_n = P_{n-1} + \alpha (T_n - T_s) + \gamma (T_n - T_{n-1}) , \quad (2-1)$$

where $T_n$ is the measured temperature, $T_s$ is the temperature set point, and $\alpha$ and $\gamma$ are adjustable regulation constants. The temperatures are computed from an empirical calibration of the resistance of the ther-
mistrors as a function of temperature. The thermistor’s main temperature
dependence is an approximately 5% change in resistance with each degree
change in temperature. However the relation between \( T \) and \( R \) is not
purely exponential and it was found that the function

\[
R(T) = R_0 e^{\Delta T(a+b \Delta T+c \Delta T^2)/T}
\]

\[
\Delta T = T - T_0,
\]

(2-2)

fits over a wide range of temperatures. The temperature \( T \) is measured in
Kelvin and, \( T_0 \) is an arbitrarily chosen reference temperature set to be 25
° C = 298.15 K. The parameters \( R_0 \), \( a \), \( b \), and \( c \) are the fitted calibration
coefficients. Typically the calibration data can not determine \( c \) with errors
smaller than its value so it was set identically to 0. The thermistors are cal-
ibrated against a reference thermistor which has been separately calibrated
against a platinum resistance thermometer (PRT). The PRT has a calibra-
tion traceable to the National Institute of Standards and Technology
(NIST). The bath and bottom plate thermistors are calibrated against the
reference thermistor by inserting the reference thermistor into the cooling
water bath. The bath is then regulated to a constant temperature and the
system is left for 2 hours to equilibrate. The resistance of the thermistors
and the bath temperature (from the calibration source) are then mea-
sured. The results of the calibration of the reference thermistor against the
platinum resistance thermometer are shown in Table 2-1. The \( \text{rms} \) resid-
ual of the fit is 4.7 \( \Omega \). Similar residuals are found when the experiment
Table 2-1. Calibration Coefficients for Reference Thermistor

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(-13.858 \pm 0.002)</td>
</tr>
<tr>
<td>(b)</td>
<td>(-0.0055 \pm 0.0001 K^{-1})</td>
</tr>
<tr>
<td>(c)</td>
<td>(0 K^{-2}) (held constant)</td>
</tr>
<tr>
<td>(R_0)</td>
<td>(110.056 \pm 0.003 k\Omega)</td>
</tr>
<tr>
<td>(T_0)</td>
<td>(298.15^\circ K) (held constant)</td>
</tr>
</tbody>
</table>

thermists are calibrated against the reference thermistor. This implies that for temperature differences around 25°C (where the resistance is approximately 100 kΩ) one has an accuracy of about 1mK. Once the calibration function \(R(T)\) is known, it may be numerically inverted to produce \(T(R)\). Since the function is monotonic over the range of interest, the numerical inversion is unique and was computed using a Brent's method zero finder [15].

Once the appropriate heater power had been computed, the temperature servo programed the heater power supply via a 12 bit (4096 steps) digital to analog converter (DAC). The DAC generates a voltage between 0 and 5V which was used as the voltage program input of an HP model 6024A DC power supply which drives the heater [16]. The power supply, in response to the voltage program, produces a voltage between 0 and 60 V. Although the power supply will remain in regulation when delivering up
to 200 Watts into the load, the temperature of the circulator bath was set so that the regulation heater typically supplies about 30 Watts.

The heater was constructed of 75 feet of teflon coated #22 gauge copper wire. The wire was coiled and placed inside the tubing near the inlet of the apparatus. The coiled wire had a large surface area which provided good thermal contact with the flowing water and the teflon electrically insulated the wire from the water. Wires with other coatings have been tried but with much less success. Apparently, only the teflon coated wire is sufficiently waterproof to prevent water from diffusing through the coating to the copper after several years of operation. The heater had a resistance of approximately 8 Ω.

2.3 Sample Cell Design

The goals of the cell design were to achieve a long narrow channel while maintaining Boussinesq convection. The cell must be narrow enough to insure that the rolls align perpendicular to the walls. If the cell is too wide the rolls will align parallel to the flow, and the system will not display a convective instability. For this reason I decided to use a depth to width aspect ratio of 1:2. Ideally one would like a long rectangular cell, however the apparatus designed in our labs are all based on a cylindrical cell. I decided to leverage the expertise in cell design accumulated by my
research group and use a cylindrical top and bottom plate. In retrospect, this may have been a mistake. The top and bottom plates are formed out of optically flat single crystal sapphire. Some of the early work was performed on cells with a silver bottom plate, however the silver proved to be too soft and fragile. The final data were all taken with the sapphire bottom plate. The reason for this choice is that sapphire is a good heat conductor (0.35 Watt/cm-K), is mechanically rigid (Young’s modulus is 5.2 \times 10^{12} \text{ dynes/cm}^2), and can be optically surfaced at a reasonable cost [17]. In addition, it is chemically inert and virtually impervious to scratches. Even better than sapphire would be the use of a single crystal of isotopically pure C^{13} diamond which has a thermal conductivity even higher than copper. Additionally it is very rigid and has good chemical resistance. Sadly, a supplier of 4 inch diamond blanks could not be found.

The physical dimensions are limited by the size of the sapphire top plate one can obtain at a reasonable cost. The cell uses a 10.16 cm diameter 0.978 cm thick cylindrical sapphire plate for both the top and bottom of the cell. The cost of each plate was approximately $500 (roughly $300 for the material and $200 for the preparation of the optical surfaces). The sapphire surfaces, before installation in the apparatus, are flat to less than 0.1 \mu m. The rigidity of sapphire is important since the sidewalls have o-rings on the top and bottom surfaces to seal the fluid inside the cell against the air in the can. The seals are formed by pressing o-rings against
the optically flat surfaces and it is important that the surfaces deform as little as possible when clamped. Indeed, the perfection of the surfaces is quite dramatically spoiled when the cell is clamped together. After assembly the parallelism of the plates can be measured optically (See Cell Assembly Procedure on page 69.). The uniformity of the thickness of the cell is then about 3 μm.

In order to achieve a large aspect ratio cell contained within a 4 inch circle, the channel is bent into a long spiral. Previous experiments on annular cells suggest that the curvature plays only a minor role in the convection dynamics [18, 19]. In the analysis of the data, it will be assumed that the arc length along the centerline of the cell is the distance parameter x, used in the theory. In the absence of any theoretical guidance this is a reasonable assumption, but could quantitatively affect the comparison of the data to the theory. The failure of the theory to include the effects of the lateral sidewalls is likely a more important omission.

The sample fluid flowed from the upper reservoir through a tube imbedded in the cooling water bath. This tube was long enough that the water was thermalized to the top plate temperature. The tube passed through the air space within the can and connected to a hole in the bottom plate. The fluid, therefore, entered the cell at some unknown temperature which is somewhere between the top and bottom plate temperatures. The
inner part of the spiral was used to thermalize the fluid and insure that it had the linear temperature profile of the conduction state. Whether this was achieved can be easily estimated. The inlet channel dimensions are 28.5 cm: 0.324 cm: 0.409 cm (length:width:height). At the highest flow rates of approximately 0.5 ml/min, the fluid spends about 7.5 min in the inner channel. The vertical thermal diffusion time $d^2/\kappa$ for the cell at a mean temperature of 26°C is 71 seconds, so the fluid spends at least 6.4 thermal diffusion times in the inlet. This is sufficient to guarantee that the fluid is well thermalized as it enters the active region of the cell. This is an estimate of the worst case. Most data was taken with flow rates substantially slower than 0.5 ml/min.

2.3.1 Nozzle Design

One would like to insure that the sidewall generates a smooth ramp of order parameter $\varepsilon \equiv \Delta T/\Delta T_c - 1$ at both ends of the active region. One could imagine several ways of achieving this; vary the thickness of the cell, vary the temperature difference, etc. The method used was based on the observation that the critical temperature difference is a function of the cell width. The width is characterized by the lateral aspect ratio $\beta$, defined to be the ratio of the width to the depth of the cell. Qualitatively one expects the zero velocity boundary conditions on the sidewall to suppress the onset of convection by contributing to the viscous drag. This has been
studied numerically by Frick and Clever [20]. A fit to their results shows that the variation of the critical Rayleigh number $Ra_c$, with aspect ratio is

$$Ra_c(\beta) = 1707.76 + 684.23 \beta^{-1.78}.$$  \hspace{1cm} (2-3)

This result is applicable when there is no applied through flow and the rolls are aligned perpendicular to the walls. One may use it to design a channel width which has any desired $\varepsilon$ profile. This method has the advantage that the plate spacing remains constant so that the channel has a constant temperature difference along its length, and therefore a constant Rayleigh number. Varying the width causes a ramp in the critical Rayleigh number (and $\varepsilon$). The temperature profile in the non-convecting state remains everywhere $T(z) = T_b - (T_b - T_t) z/d$, where $T_b$ is the bottom plate temperature, $T_t$ is the top plate temperature, $z$ is the vertical distance, and $d$ is the thickness of the cell.

The channel used in the experiment was designed to have $\varepsilon$ match smoothly between the value in the inlet, $\varepsilon_1$, and that of the active region, $\varepsilon_2$. A mirror of this ramp is used at the outlet. The ramp was chosen to have continuous first and second derivatives at both ends of the ramp. If we let $\varepsilon(x)$ be the stress parameter in the ramp, and the length of the ramp be $l$, we have the criteria
\[\varepsilon(l/2) = \varepsilon_2, \quad \varepsilon(-l/2) = \varepsilon_1, \quad \varepsilon'(l/2) = \varepsilon'(-l/2) = 0, \quad \varepsilon''(l/2) = \varepsilon''(l/2) = 0. \quad (2-4)\]

Since there are six constraints, the simplest way to achieve them is to use a quintic polynomial for the functional form of the ramp. Solving Eq 2-4 for the coefficients of the quintic polynomial yields

\[\varepsilon(x) = \frac{1}{2} (\varepsilon_1 + \varepsilon_2) + \frac{15}{8} (\varepsilon_2 - \varepsilon_1) \frac{x}{l} - 5(\varepsilon_2 - \varepsilon_1) \frac{x^3}{l^3} + 6(\varepsilon_2 - \varepsilon_1) \frac{x^5}{l^5}. \quad (2-5)\]

This is plotted in Figure 2-4. One may now use the definition of 
\[\varepsilon = \frac{Ra}{Ra_c} - 1\] to rewrite Eq 2-5 as a ramp in the critical Rayleigh number as

\[Ra_c(x) = \left[\frac{1}{2} (Ra_{c1}^{-1} + Ra_{c2}^{-1}) + \frac{15}{8} (Ra_{c2}^{-1} - Ra_{c1}^{-1}) \frac{x}{l} - 5(Ra_{c2}^{-1} - Ra_{c1}^{-1}) \frac{x^3}{l^3} + 6(Ra_{c2}^{-1} - Ra_{c1}^{-1}) \frac{x^5}{l^5}\right]^{-1}. \quad (2-6)\]

It is important to note that this expression is independent of Ra and therefore the applied temperature difference. It depends only on the widths of the two regions being connected. Indeed one may use Eq 2-3 to rewrite Eq 2-6 in terms of the two widths. The reader will be spared the resulting expression since it is not particularly revealing. The conclusion,
FIGURE 2-4. Critical Rayleigh Number in Inlet and Outlet Nozzles
This figure shows the variation of the $\varepsilon$ used in the nozzle which connects the inlet region to the active region of the cell. A quintic polynomial is used so that one has a curve which matches smoothly (continuous first and second derivatives) with the straight sections of the inlet and active regions of the channel.

however, is that a single geometry provides the same smooth ramp in $\varepsilon$ for all applied temperature differences.

The final channel design uses an inlet width of $\beta_{\text{inlet}} = 1$, a straight section width of $\beta_{\text{active}} = 2$, and a ramp length of $l = 4$ (all lengths are scaled by the thickness). The resulting design is shown in Figure 2-5.

Unfortunately in the manufacturing of the part, the sidewall material proved to be very difficult to cut accurately (See A Search for a Better Sidewall Material on page 59.) The channel width in the final part was wid-
FIGURE 2-5. Width of Channel in Inlet Nozzle

This figure shows the channel width in the ramps. The lower curve shows the design which generates the smooth ε profile shown in Figure 2-4. The upper curve was the width profile which was achieved in the machined part.

ened uniformly by 0.095 cm so the ε profile shown in Figure 2.3 is not exactly achieved. Robert Burns succinctly put it: "The best laid schemes o' mice an' men Gang aft a-gley." The final dimensions are summarized in Table 2-2. A top view of the cell geometry is shown in Figure 2-6.

2.3.2 Sidewall Composition and Thermal Modeling

A prototype sidewall was built from Delrin. The convection which was observed in this channel was nonuniform in amplitude. The observed

1. Delrin® is a brand name from Dow Chemicals for acetyl-homopolymer.
Table 2-2. Cell Dimensions

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness</td>
<td>$d$</td>
<td>0.3234±0.001 cm</td>
</tr>
<tr>
<td>Inlet width</td>
<td>$\beta_{\text{inlet}}$</td>
<td>1.264±0.002 $d$</td>
</tr>
<tr>
<td>Channel width</td>
<td>$\beta_{\text{active}}$</td>
<td>2.245±0.001 $d$</td>
</tr>
<tr>
<td>Ramp Length</td>
<td>$l$</td>
<td>3.925±0.005 $d$</td>
</tr>
<tr>
<td>Active Region</td>
<td>$x$</td>
<td>59.0±0.2 $d$</td>
</tr>
</tbody>
</table>

nonuniformity was consistent with an epsilon variation of about 1% around the cell. Moreover, the convection was strongest near the outlet and weakest in the interior. One possible explanation is based on the difference of the thermal conductivity between the water and the sidewall material. Delrin has a thermal conductivity of about $2/3$ that of water. Since there is nominally a uniform power per unit area flowing through the cell, there is a larger temperature difference across the sidewall. Portions of the cell which have a greater fraction of sidewall area to cell area are therefore "hot spots" as is shown in Figure 2-6. This leads to lateral thermal gradients, which unlike vertical gradients, have no critical value. They always induce flows; therefore it is very important that they be minimized. Additionally, the varying temperature differences along the channel make epsilon nonuniform (in the active section) making comparison to theory problematic. The cell design is not azimuthally symmetric so there is a small large scale azimuthal gradient in addition to a radial gradi-
Region of reduced heat transport due to a mismatch between the sidewall and the fluid conductivities.

FIGURE 2-6. Thermal Problems Due to the Sidewall Shape
This figure shows that the spiral shape of the cell leads to a large region at the end of the channel which is solid material. If the conductivity of the sidewall material is lower than the working fluid, this area will create a hot spot.

ent. The radial gradient isn't too much of a problem since the radius of the center of the channel only changes by about 20%. However the azimuthal gradient causes a nonuniformity in the temperature difference along the channel. To determine whether this gradient is a serious problem, thermal modeling of the cell was performed. The thermal modeling program, thermit, uses a simple 2-D relaxation method which is quantitatively accurate for modeling long slabs of material. (See APPENDIX E THERMIT - 2D Thermal Modeling on page 266.) This program was used to model a cross section of the cell since the region of interest (the wide portion of the channel) is at the outside of the disk where the curva-
ture of the slabs is small. The model is presumably nearly applicable. The approximation is tantamount to assuming that there is no azimuthal heat flow. The results near the center of the disk however, will not be quantitatively accurate.

To estimate the magnitude of the azimuthal temperature gradient two extreme cases are compared. Case one consists of a cross section taken through the cell at the beginning of the active region of the channel. Case two consists of a cross section taken at the end of the active region. These two cross sections are shown in Figure 2-7.

The boundary conditions used are a constant temperature $T = 0$ on the top of the top sapphire and all other external boundaries are assumed to be insulating. The heater element generates a constant power per unit area. The entire cell is simulated and the results are normalized so that there is a unit temperature difference along the line through the center of the cell. Therefore the temperature differences plotted can be interpreted as a percentage of the total temperature difference across the cell assembly. The thermit configuration file used for the modeling near the inlet is shown in Table 2-3.

The temperature difference across the fluid in the cell is shown in Figure 2-8. About 83% of the temperature difference across the cell assembly is actually across the fluid layer. The remaining 17% of the tem-
FIGURE 2-7. Two Cross Sections Used in Thermal Modeling

This figure shows the lines along which the thermal modeling was performed. The solid line indicates the cut across the inlet section (case 1). The dashed line shows the cut across the outlet (case 2). The bottom portion of the figure shows a side view of the inlet cross section. Each rectangular region was included in the thermal modeling. The o-rings used to seal the cell are not included in the simulation.

Temperature difference is across the top and bottom sapphires. The tempera-
# Simulation of cell near inlet - lengths are in inches
scale 4 1.025
region 0 0 4 .4 .003 # Bottom Plastic
region 0 .4 4 .65 35 # Bottom Sapphire
region 0.2000 .65 0.5375 .775 .04 # Plastic in Sidewall
region 0.5375 .65 0.7875 .775 .006 # Water in Cell
region 0.7875 .65 0.9750 .775 .004 # Plastic in Sidewall
region 0.9750 .65 1.1000 .775 .006 # Water in Cell
region 1.1000 .65 1.3500 .775 .004 # Plastic in Sidewall
region 1.3500 .65 1.4750 .775 .006 # Water in Cell
region 1.4750 .65 1.7250 .775 .004 # Plastic in Sidewall
region 1.7250 .65 2.1250 .775 .006 # Water in Cell
region 2.1250 .65 2.3375 .775 .004 # Plastic in Sidewall
region 2.3375 .65 2.4626 .775 .006 # Water in Cell
region 2.4626 .65 2.7125 .775 .004 # Plastic in Sidewall
region 2.7125 .65 2.8375 .775 .006 # Water in Cell
region 2.8375 .65 3.0250 .775 .004 # Plastic in Sidewall
region 3.0250 .65 3.2750 .775 .006 # Water in Cell
region 3.2750 .65 3.8000 .775 .004 # Plastic in Sidewall
region 0 .775 .4 1.025 .35 # Top sapphire
insul 4 0 4 1.025 # Right side
insul 3.8 .65 .4 .775
# Air Gap on right side
insul 0 .65 .20 .775 # Air Gap on left side
insul 0 0 4 0 0 # Below Heater Plastic
insul 0 0 0 1.025 # Right side
temp 0 1.025 4 1.025 0 # Top of sapphire
power .225 .4 3.775 .4 1 # Heater Element
init 54
end

Table 2-3. Configuration File Used for Thermal Modeling near Inlet

temperature difference across the inlet and outlet portions of the channel differ by
0.8%. It is this variation which should be made as small as possible.

2.4 A Search for a Better Sidewall Material

The observed nonuniformity in ε agrees qualitatively with the thermal
modelling. The problem is at the 1% level. The temperature control and
FIGURE 2-8. Temperature Difference Across Cell from Thermal Modeling

This figure shows the computed temperature difference across the cell for the two slices shown in Figure 2-7. The ordinate is the temperature difference across the water/sidewall layer which has been normalized to unity across the center of the cell assembly. The rest of the temperature drop is across the top and bottom sapphires. There is a difference of 0.8% between the temperature difference at the inlet and the outlet of the cell. Additionally the inlet slice is much more uniform than the outlet slice.

Geometrical perfection of the cell introduce a nonuniformity which should be much smaller. The homogeneity can be (and has been) improved by adjusting the size of the heater relative to the cell, but the problem would be mostly eliminated if one could find a sidewall material which has a better thermal match to water. A search of the literature revealed that most plastics have thermal conductivities about 1/3 that of water. Indeed, delrin is anomalously large in that its conductivity is about
2/3 that of water. However, one can get composite materials which have very large thermal conductivities. These materials are made by mixing epoxy with a high conductivity material such as silica powder or colloidal metals (often gold or silver or brass). I eschewed the colloidal brass and silver resins for fear that electrochemistry might contaminate the working fluid. The colloidal gold epoxies were avoided as they are quite expensive. This left the alumina filled resins, which as it turned out, are difficult to machine.

I engineered a composite epoxy with the same conductivity as water. This was performed by mixing two epoxies together and casting them into a blank from which the sidewall was later machined. The two epoxies were Epotek 905 which is a high conductivity alumina filled resin and Epotek 509F which is low conductivity unfilled resin. The thermal conductivity of the composite can be tuned by simply changing the mixing ratio. These two epoxies were chosen for their chemical compatibility with each other and for their resistance to degradation in water. It was later discovered that the 509F resin is, in fact, the binder used in the 905 material. That suggests that the process would have simplified by using the pure 509F and mixing in a measured amount of silica powder. “Hindsight is 20-20”.
To determine the correct mixing ratio, the individual materials were first characterized. If the recipe described below, is to be repeated, one must be aware that the characterization is only accurate for the particular batch of 905 material used. The reproducibility of the results is therefore limited by the reproducibility of the component material. The 509F epoxy is a pure material so this characterization is probably accurate for future batches as well.

To measure the thermal conductivity one must accurately measure the heat transported through a known thickness of material. This was done by casting blanks of the two materials and machining them into disks. The components of the epoxies were mixed as per the mixing specification. Then the liquid was put into a vacuum desiccator and exposed to a vacuum. This removes the dissolved gasses and small bubbles. This is necessary since the liquid epoxy is very viscous and traps many small bubbles which could affect the final conductivity. In the process of pumping, the material bubbled up into a large froth before the bubbles broke. Since the bubbles don’t suspend the alumina particles as well as the bulk material, some sedimentation of the alumina particles likely occurred. The desiccator vacuum was released and the material was left to cure overnight in the desiccator. The blank was then released from the plastic mixing container and baked in the oven at 65 °C for another night to insure that the material had completely cured. Then the material was fly cut into disks. The
material proved to be very difficult to machine since the alumina particles rapidly dull the cutting tool. A carbide tool was used, and it had to be re-sharpened many times. The pure resin machines very easily. In the future, this procedure would probably be much simpler if one used the colloidal metals as the high conductivity component.

Once the disks were machined they were put into the cell in the configuration shown in Figure 2-9. The sample disk was put inside of a spacer

![Diagram](image)

**FIGURE 2-9. Configuration for Measuring Thermal Conductivity of Epoxies**

This figure shows the sidewall configuration for measuring the thermal conductivity of the epoxy sidewall materials. There is a spacer ring around the outside, and the sample material in the form of a disk fills the interior.

ring constructed out of the unfilled epoxy. A thin layer of vacuum grease was put on the top and bottom of the sample disk before it was clamped between the sapphires. This eliminated the thin air gap which would have formed an insulating layer. Indeed this worked so well at eliminating the
air gap that it was very difficult to disassemble the cell after the measure-
ments were performed!

Once assembled, the temperature regulation servos were set to a desired
temperature and the apparatus was left for 3000 seconds to equilibrate.
Then the power in the bottom plate servo was averaged for 5 minutes.
The process was then repeated for a new temperature difference. The rea-
son that the measurement was done at constant temperature rather than
constant power is that the apparatus equilibrates much more rapidly at
constant temperature. Once equilibrated, the servo power is quite con-
stant. (It typically alternates between two adjacent levels on the DAC). A
typical data set is shown in Figure 2-10. It shows the power dissipated in
the bottom plate heater as a function of the temperature difference
\[ \Delta T = T_{\text{bottom}} - T_{\text{bath}}. \]
The data was taken in steps of 0.1 °C up to a \( \Delta T \)
of 1.0 °C. The mean temperature of the cell was held fixed at 26 °C by
symmetrically changing both the bath and bottom plate temperatures.
The linear fits show that the conductivity is constant over this tempe-
ration range. The conductance of the sample is defined as \( C = P/\Delta T \), how-
ever to eliminate small discrepancies between the calibrations of the
thermistors, the slope of a fit of \( P \) vs. \( T \) is used in the calculations. Mea-
surements in four configurations were performed: the spacer ring with the
filled epoxy disk, the spacer ring with the unfilled epoxy disk, the spacer
ring filled with water, and the spacer ring with only air in the cell. The last

64
FIGURE 2-10. Conductivity of Filled Epoxy.
This image shows the data sets for the conductance of the various samples. This cell consisted of a ring of unfilled epoxy (509F) surrounding either: a disk of filled epoxy, a disk of unfilled epoxy, water, or air). The conductance of the entire cell is given by the slope of the line. The quality of the fits and the zero intercepts indicate that the conductivity of the samples was effectively constant over the measured temperature range and that the bath and bottom plate thermistor calibrations agree. The mean temperature was fixed at 26°C.

configuration allows one to measure the stray conductance through the parts of the cell other than the sample. The configuration with the spacer ring filled with water was done as a check of the procedure; it should yield the textbook value for the conductivity. The results are summarized in Table 2-4. With this data one may now compute the conductivity of the various materials using the relations

65
<table>
<thead>
<tr>
<th>Material</th>
<th>Area of sample (cm²)</th>
<th>Thickness (cm)</th>
<th>Conductance (Watt/°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ring + Air</td>
<td>60.4±0.1</td>
<td>0.321±0.001</td>
<td>0.491</td>
</tr>
<tr>
<td>Ring + 509F</td>
<td>56.88±0.03</td>
<td>0.321±0.001</td>
<td>0.751</td>
</tr>
<tr>
<td>Ring + 905</td>
<td>56.93±0.03</td>
<td>0.320±0.001</td>
<td>1.696</td>
</tr>
<tr>
<td>Ring + Water</td>
<td>60.4±0.1</td>
<td>0.321±0.001</td>
<td>1.568</td>
</tr>
</tbody>
</table>

\[ \lambda_{\text{eff}} - \lambda_{\text{air}} = \left( C - C_{\text{ring}} \right) \frac{d}{A} \]
\[ \lambda = \lambda_{\text{eff}} \left(1 - \frac{\lambda_{\text{eff}} d_s}{\lambda_s d} \right)^{-1}, \]  

(2-7)

where \( \lambda \) is the conductivity of the pure sample material, \( \lambda_{\text{eff}} \) is the effective conductivity of the combination of the sample and the top sapphire, \( \lambda_{\text{air}} \) is the conductivity of the air displaced by the sample cell, \( C \) is the measured conductance of the sample, \( C_{\text{ring}} \) is the conductance of the empty (air filled) ring, \( d \) is the thickness of the sample, \( d_s \) is the thickness of the sapphire, and \( A \) is the cross sectional area of the sample. The first line of Eq 2-7 computes the effective conductivity of the sample and sapphire combination and the second line corrects for the temperature drop across the sapphire. The temperature drop across the aluminium bottom plate can safely be ignored since the conductivity of the aluminium is much larger than the sample. From standard tables one finds the conductivities of air and sapphire at 26°C to be \( \lambda_{\text{air}} = 2.6 \times 10^{-4} \) Watt/(cm °C) and \( \lambda_s = 0.385 \) Watt/(cm °C) [17]. Applying Eq 2-7 to the data in
Table 2-4 gives the conductivities of the materials. The result for water differs by 3% from the value listed in the literature [17]. There are probably some systematic errors in the experiment which would account of the discrepancy. However the purpose of this calculation was to engineer a material with the same conductivity as water. For this reason the mixing ratio was chosen to reproduce the experimentally measured water conductivity (as opposed to the CRC value). I further assume that the conductivity will vary linearly between the two values as one mixes them so that the conductivity should, to lowest order be

\[ \lambda_{\text{mix}} = x \lambda_{\text{filled}} + (1 - x) \lambda_{\text{unfilled}}, \]  

(2-8)

where \( x \) is the mixing ratio. Therefore one has a mixing ratio of \( x = 0.795 \). A new disk was constructed with this mixing ratio and it was characterized using the same procedure. The new material has a measured conductivity of \( \lambda_{\text{mix}} = 6.29 \text{ mWatt}/(\text{cm } \circ C) \) which agrees with the mea-
sured value of water to within 0.3%. Clearly the procedure works as expected. This material was then used to construct the sidewall.

2.4.1 Perturbation Heater

It is sometimes desirable to force convection rolls in the apparatus. (See *Measuring the Critical Temperature Difference as a Function of Reynolds Number* on page 138.) To this end, a small heater was mounted in the inlet of the active region of the cell. The heater consists of an approximately 2 cm long piece of 0.013 cm diameter nichrome wire. It was threaded through a set of parallel holes drilled in the sidewall as shown in Figure 2-11. The wire is pressed against the side of the channel and does

![Perturbation Heater](image)

**FIGURE 2-11. Blow Up of Inlet Section of Sidewall Showing the Perturbation Heater**

A perturbation heater is mounted at the inlet of the channel. It locally heats the inlet sidewall and the nearby fluid to drive convection rolls.
not perturb the channel flow appreciably. It has a resistance of 3.0 Ω and it is typically used to dissipate about 50 mW. Since the heater is threaded through the sidewall, most of the heat dissipated locally warms the sidewall. It is typically used by applying a sinusoidally varying voltage with a Wavetek Model 178 function generator [21]. Since the way it couples to the convective flow is complicated, the amplitude of the drive is kept constant in a set of measurements, but the frequency is varied.

2.4.2 Cell Assembly Procedure

To assemble the cell one first cleans and dries the components. The cleaning procedure is not too critical, however, if the cell surfaces are clean, one tends to have fewer bubbles. I clean the top and bottom sapphires with soap (Lemon Joy) and water. I found that it is much easier to rinse the soap off with the Goleta tap water. The tap water has enough dissolved minerals that it cleans the soap off of the surface. Next rinse thoroughly with deionized water. This cleans the Goleta water off of the surface. Rinse the surface with methanol and blow it dry with dry nitrogen. It is best to work under a laminar flow hood when drying the surface to keep it relatively dust free. I use an analogous cleaning procedure for the sidewalls. The only additional restriction is that the epoxy gets soft when heated to approximately 40 °C so care must be taken to keep the material cool while washing. The sidewall material is not rigid enough to hold its
shape accurately by itself. This problem is compounded by the fact that the sidewall design creates a large flat spiral spring. To prevent this spring from unwinding or distorting, the sidewall material is held in place by a series of pin holes drilled through the sidewall. Corresponding holes are drilled into the bottom plate. The holes are nominally 0.034 inches in diameter. The holes are filled with pins machined from delrin. This material was chosen since it is relatively strong, easily machinable and has a thermal conductivity close to that of water. In addition, prototype cell sidewalls were constructed entirely of delrin and I had a large stock of extra pins. Once the sidewall is pinned in place, one may measure the uniformity with one of two procedures. The simplest and most accurate method is to use a series of stainless steel pins of increasing diameter. Successively larger pins are inserted in the channel until they just touch the inside and outside of the channel. The pin diameter is then measured with a micrometer. This can be repeated at several locations around the channel to get an estimate of the uniformity of the channel width. I found that the channel width was uniform to approximately 0.003 in. This is about as good as one may achieve with the epoxy sidewalls because the epoxy is a composite. The suspended alumina particles are quite large. Inspection of the material under a microscope shows that they are at least 15-20 microns in diameter. This causes the surfaces of the final part to be rough and prevents one from machining them to much better tolerance
than was achieved. In addition, the aforementioned problem with the sensitivity to temperature means that the part can change shape while being handled. However, since the material was machined while pinned to a machining surface using the same pin pattern, it returns quite reproducibly to the same shape when installed into the cell.

The reproducibility of the sidewall shape upon subsequent re-assemblies of the cell seems limited by the quality of the holes in both the sapphire and the sidewall. The holes in the sapphire vary by about 0.002 inch in diameter, ranging in diameter from 0.034 inch to 0.036 inch. They were intended to be 0.032 inch. The pin hole centers are located with an average error of 0.0014 inch. The x-y locations of the hole centers were measured with a traveling microscope. I then wrote a simple fitting program which minimized the least square distance between the measured coordinates and the specified locations. The function which was minimized was

$$\chi^2 = \sum_{i=0}^{N} |X_i - \alpha R_\theta (x_i + x_0)|^2 ,$$  \hspace{1cm} (2-9)

where $X_i$ is a list of 2D vectors of the desired pin locations, $x_i$ is the list of measured coordinates, and $N$ is the number of points. $R_\theta$ is the 2D rotation matrix for rotation by $\theta$. The fit parameters are $\alpha$, which is the overall scale calibration of the travelling microscope, $\theta$ which is the rotation angle, and $x_0$, which is the coordinates of the origin. If both the
numerical mill used to machine the part and the traveling microscope used to measure it are properly calibrated, α should be unity. Both θ and x₀ are arbitrary and depend only on the placement of the part on the platen of the microscope. The error in these parameters, however, is not arbitrary, but measures the accuracy of the pin location measurements. The results of the fit are summarized in Table 2-6 and the residuals (distance between desired location and actual location) are listed in Residuals of Fit to Retaining Pin Locations on page 262. The average residual is only 0.0014 inches. Since the sidewall is pinned to these holes it is not surprising that the channel width varies by several mil. One finds that alpha is unity to within the error of the measurement, which indicates that the microscope and mill distance scale are well calibrated with respect to each other.

One may wonder why the accuracy of the hole positions in the sapphire is so poor. The reason for this variation is that sapphire is a very difficult
material to machine. The pin holes were ground into the surface with small diameter sintered diamond grinding rods. During machining, the rods wore causing the hole diameters to be nonuniform (and somewhat conical in profile). In three holes the grinding rod broke and embedded itself within the hole. This remaining material was later removed by immersing the sapphire in *aqua regia* so that the metal matrix holding the diamond crystals was etched away. This cleared the holes, but the holes left were not as deep as desired and probably did not hold the pins firmly in place. Also, the delrin pins, while quite rigid for plastic, do deform with use as do the pin holes drilled in the sidewall. Overall the cell uniformity which was achieved was probably as good as one may expect for such a soft material.

After the channel uniformity was characterized, the cell was assembled by inserting the o-rings and lightly clamping the top and bottom plates together with the bottom plate screws. The cell was then positioned under an expanded laser beam as shown in Figure 2-12. The top and bottom surfaces of the cell act as the plates of an optical interferometer. On the screen a series of fringes appear with a spacing which corresponds to the plate spacing varying by $\lambda/2$. Before adjusting the bottom plate screws, the beam expanded laser is made perpendicular to the top plate surface by observing the light which reflects from the top surface of the top sapphire. When properly aligned, the light retraces its path and refo-
FIGURE 2-12. Experimental Setup for Measuring the Cell Spacing Uniformity

cuses onto the focal point of the objective lens. A small screen is intro-
duced near the focal point and the laser and lens assembly are aligned relative to the cell until the reflected spot is coincident with the spot produced by the light exiting the laser. Once aligned, the small screen at the focal point is removed and the fringe pattern is then observed on the large screen shown in Figure 2-12. There are a large number of spurious fringes which are generated by reflections from each interface and dust in the optics. However, the spurious fringes do not move when the adjustment screws are tightened. It is the dynamic aspect of the fringes which are generated from the cell interior, which allows the operator to distinguish them from the spurious fringes. He-Ne laser light has a wavelength of $\lambda = 0.633\,\mu m$, so each fringe shows a variation of $0.317\,\mu m$. The bottom plate screws are tightened to compress the sidewall o-rings while simultaneously attempting to minimize the number of fringes.

Unfortunately, these two objectives are mutually antagonistic. One would like as small an o-ring compression as is possible and still insure a reliable seal. As one increases the o-ring compression, the clamping pressure increases and the force that the o-rings exert on the sapphires causes them to bend. The flow distributor, which presses on the top sapphire, is designed to support it in line with the o-ring groove. Similarly the bottom plate clamping ring presses on the same circle.
The best uniformity I have been able to achieve, while insuring that the cell is sealed, is approximately 10 radial fringes from center to the edge of the cell. After the cell had been running for several months, this number increased to 20 fringes, of which approximately 10 cross the active region of the cell. With a circular cell, and a similar clamping arrangement, one may do much better. Others have been able to achieve as few as 2 or 3 fringes. The reason I have been unable to do better is almost certainly due to the complicated cell geometry. This cell has a very large surface area in contact with the top and bottom plates. The sidewall thickness uniformity is quite good, varying by only $\pm 10^{-4}$ inches, however to prevent fluid from flowing between the channels of the spiral, the top and bottom surfaces must be clamped tightly against the sapphires. I suspect that some of the cells which are much flatter, are not touching the sapphire uniformly, but are riding on the o-ring.

Before filling, the cell is flattened, but the screws are not tightened until the sapphires touch the sidewall. The cell is only tightened enough to make the o-rings seal. This allows the sample fluid to wet all of the sidewall surfaces. The sample fluid is water which has been filtered, distilled and de-ionized using the Millipore milli-q filtering system. It is then degassed by using a water aspirator vacuum system connected to one of the building water taps as shown in Figure 2-13. There is an extra over-
Before filling the cell, the sample fluid is degassed with the simple setup shown. A water aspirator is connected to a filter flask holding the sample fluid. An overflow flask is inserted between the two to prevent contamination of the sample.

flow flask inserted in line with the vacuum system to prevent any fluid which may leak from the aspirator from contaminating the sample fluid.

The fluid is degassed only to aid in eliminating bubbles during the filling phase. If the fluid is sufficiently degassed, small bubbles which get trapped in the cell or against the sidewall will dissolve. The pumping system, however, aerates the fluid, so the experiment is performed with fluid which is effectively saturated with air.

The cell is filled with degassed water by connecting a syringe to the inlet. The cell is slowly filled and rocked back and forth to help free trapped bubbles. When the cell is completely filled, the outlet is plugged and the syringe plunger is pulled back to create a weak vacuum in the cell. This causes air which is trapped in the pin holes, the o-ring grooves, and
between the sidewall and the sapphire, to expand into the channel. The cell is then rocked until the air can be eliminated through the exit hole. This process is repeated several times. There will, however, still be some air trapped against the side walls. The inlet and outlet hoses are then connected to the flow system and the sample fluid is flowed through the cell. I have found that any bubbles which are trapped in the cell will persist for a very long time; presumably since the flowing sample fluid is saturated with dissolved gasses. A simple trick, however, allows one to eliminate the remaining bubbles. After the system is assembled, and the cooling water bath is on, heater tape is wrapped around the upper reservoir flask and its temperature is raised to about 35°C. Additionally, the bath temperature can be reduced to about 20°C. The idea is that the solubility of gas is reduced at higher temperature. The water in the upper reservoir is saturated at 35°C, but is not saturated at the lower cell temperature. This causes the bubbles to slowly dissolve. The system is then left overnight (or a couple of days) until all of the remaining bubbles have vanished. The upper reservoir heater can then be turned off. If the cell is not leaking, the bubbles will not reappear.

Once the cell is connected to the flow system, the bottom plate heater is connected and the bottom plate temperature regulation loop is started. The bottom plate is regulated to 26°C which is near to the mean operating temperature. The cell is re-flattened under the beam expanded laser.
This way the cell is subjected to the same hydrostatic pressure (which comes from the height difference between the lower reservoir and the cell) and mean temperature as it will be when data are taken. This procedure minimizes the thermal stresses to which the cell is subjected. Additionally, since the bottom reservoir is never raised or lowered, the hydrostatic pressure in the cell does not change. This is the reason that the flow impedance is mounted between the upper reservoir and the cell. It prevents the pressure in the cell from changing when the flow rate is changed. The screws are now tightened until the sidewall is pressed securely against the top and bottom plates. The thickness of the cell is then measured.

To measure the cell thickness, the cell is placed upside down on a stainless steel spacer ring which presses against the top sapphire. The spacer ring thickness is constant to $\pm 5 \times 10^{-5}$ inches. The assembly is placed on a flat granite table. The thickness of the entire assembled cell is measured with a height gauge. The height gauge is used to compare the cell measurements to a stack of calibrated Johansen blocks. The cell is measured at 4 locations around the circumference. The height gauge is left at a fixed location on the granite table and the cell and spacer ring are rotated as a unit under the height gauge. The granite table surface is quite flat, but this procedure minimizes error which would arise from moving the height gauge over the surface. If the surfaces are kept clean, this measurement is reproducible to about $\pm 2 \times 10^{-4}$ inches. The cell thickness is then deter-
mined by subtracting the thickness of the individual components which are measured by a micrometer. The overall thickness of the cell was found to be $0.1274 \pm 0.0003$ inches.

The cell assembly is mounted in the can and a small bag of desiccant is placed in the airspace. This keeps the air around the cell dry. The rest of the airspace in the can is filled with bubble paper and open cell foam. This prevents the air in the can from convecting. The can is installed in the cooling water bath and the apparatus is leveled. The leveling is performed, by placing the stainless steel spacer ring on the top of the top sapphire and centering a machinist's level on the top of the spacer ring. Adjustment screws on the feet of the apparatus are adjusted until the level bubble is centered. To eliminate errors from the calibration of the level, the spacer ring and level are rotated as a unit by $180^\circ$. The bubble in the level will typically move by one or two divisions which corresponds to 50 to 100 $\mu$ radians. The screws are adjusted so that the average of the two readings is centered in the scale. This procedure is then repeated with the level rotated to $90^\circ$ and $270^\circ$. The system is thus oriented perpendicular to g to within 50 $\mu$ radians.

The next step in the apparatus assembly is to fill the cooling water bath. The original apparatus design uses a neoprene gasket to seal the top plate of the cooling water bath to the rest of the box. It was found however,
that this gasket creeps with time causing the optics mounted on the top plate to drift unacceptably. For quick testing this gasket is still used. However, for the final data run, I sealed the top plate in place by applying a small bead of Dow Corning aquarium cement on the stainless steel surfaces and then bolting the plate in place. This method provides a more mechanically rigid mounting but has the disadvantage that the glue must be given at least 24 hours to cure. Later versions of the apparatus correct this design flaw by using an o-ring seal. The water bath is filled with deionized water and the beam expanded laser is again mounted over the cell. When the bath pump is turned on, the fringes move somewhat, but a properly clamped cell is relatively insensitive to the state of the bath pump even though the path pump exerts a substantial force downwards on the top plate. Although there is no reliable measurement of the water pressure in the cooling water bath, it can be readily estimated. If one (accidentally) opens the bubble relief valve on the cooling water bath, the column of water will reach to the ceiling. Therefore the pressure head is approximately 15 feet of water which is about 0.5 atm. Since the surface area of the top plate is 12.6 in\(^2\), the cooling water bath exerts a force of nearly 100 lbs on the sapphire. It is quite remarkable that the cell doesn’t distort more than a couple of fringes.

The next task is to remove the bubbles which get trapped in the cooling water bath during the filling. This is somewhat problematic. There is a
relief valve on the top plate which can be used to bleed off the large bubbles. However, this apparatus has many places where large bubbles can become trapped. The bath pumping system uses the fill tube as an atmospheric reference on the low pressure side of the pump. Since the bubbles are at their largest size at this point in the pump loop, they will accumulate on the fill tube and the trapped bubbles will slowly come out. Moreover, once the bath is free of bubbles, this mechanism insures that it remains so. The problem is that since a lot of air can get trapped, this process can take a very long time. I have found that one can greatly speed the process by connecting a tube between the bubble relief valve and the fill tube and adjusting the relief valve so it only slowly pours into the fill tube. This is much more efficient than simply waiting. In this configuration, the system will effectively become free of bubbles overnight. After the bath is clear, the valve is closed and data taking can begin.
2.5 The Bottom Plate Heater

2.5.1 The Heater Design

An ideal heater would consist of a uniform disk which produced a constant power per unit area. The goal of this design is to approximate this ideal with a series of wires. For the purposes of the discussion, I will assume that the diameter of the wire is much smaller than the spacing between adjacent wires so that the width of the wire can be neglected. This is reasonable since, for the heater constructed, the wire was 0.010 inches in diameter, and the spacing between the wires was 0.050 inches, so the ratio of the wire diameter to the spacing is 5:1. For the purposes of this discussion I will restrict the design to a disk of unit radius. The actual design is simply scaled up to the required size.

Consider a heater which consists of a series of concentric rings of wire. Such a heater has the obvious advantage that it preserves the azimuthal symmetry of the disk. Unfortunately, since the rings are not connected to each other, there is no way to get the current into the rings! If one were to use an inductive method to induce current, it would be difficult to insure that the same current was induced in each ring. I have modified the ring design to connect the rings to form one connected path. One can then bend a wire to this shape and run current through the wire. Alternatively, one could etch this pattern into thermofoil.
Consider the simple case of two concentric circles. To perform the connection, one cuts an arc out of the circles and uses the removed arc length to perform the connection as shown in Figure 2-14.

![Diagram of two concentric circles with an arc removed and a theta symbol](image)

**FIGURE 2-14. Heater Design for Two Rings**

This diagram shows the minimal heater design. The joint connecting the inner and outer loops is constructed so that it has the same length as the wire removed.

To insure that this modified pattern generates the same total power as the unmodified circular pattern, one has the constraint

\[(R_i + R_{i+1}) \theta_i = \pi (R_{i+1} - R_i) \]

(2-10)

The left hand side is the length of the arcs removed from the circles and the right hand side is the sum of the lengths of the two connecting arcs.
Next consider the question of how to add another ring to this system. In principle the new joint may be located anywhere on the new ring. One must have a criterion for specifying the angle from the origin to the new joint. Since each joint creates a local perturbation in the heat generation, one wishes to space them as far apart as possible. Consider two simple algorithms.

The first method is to adjust the angle to maximize the sum of the distances from the new joint to the previous joints. Choose the new angle

\[ \theta_n = \max_{\theta} \sum_{i=1}^{n-1} |x_n(\theta) - x_i|, \]  

(2-11)

where \( x_i \) is the position of the \( i \)th joint. The second method is to adjust \( \theta \) to maximize the distance from the new joint to its nearest neighbor. For each trial value of \( \theta \) one computes the distance between the new joint and all of the previous joints. The closest joint is the nearest neighbor. The nearest neighbor changes as one considers different trial values of \( \theta \). One chooses the angle which maximizes the minimum distance. Symbolically one has

\[ \theta_n = \max_{\theta} \min |x_n(\theta) - x_i|. \]  

(2-12)

Figure 2-15 shows the results of these algorithms iterated for 15 rings.
FIGURE 2-15. Two Algorithms for Adjusting the Joints

This figure shows the results of the two algorithms iterated for 15 rings. The left image shows the pattern which results from the first algorithm which maximizes the sum of the distances to the inner joints. The right image shows the pattern which results from the second algorithm. To help illustrate the difference between the two algorithms, a grey line connects the joints in the order of their creation.

2.5.2 Construction of the Bottom Plate Heater

The construction of the bottom heater is somewhat involved. Since it is crucial in the success of this experiment, its construction will be briefly discussed. The heater was constructed from a 4 inch lexan blank. The lexan was first machined into a disk with the cross sectional profile shown in Figure 2-16. Then a numerical milling machine was used to cut the line pattern of the heater using the maxi-min algorithm. The final heater had 35 rings spaced 0.050 in apart. The mill used was a 0.010 in diameter end mill to cut a groove which is just wide enough that when the wire is pressed into the groove, it will be held in place by friction. The wire used
FIGURE 2-16. Cross Section of the Heater Blank

The lip on the edges acts as a reservoir to contain a pool of epoxy which is used to seal the surface.

was 0.010 inch manganin which has a resistivity of 3 Ω/ft. The 35 rings have a total length of 16.04 ft, so the predicted resistance is 48 Ω. The actual measured resistance of the completed heater was a slight function of temperature but at 25°C it was 49.8 Ω.

Pressing the wire into the machined groove was quite straightforward except at the small arcs which form the joints. It is difficult to get the wire to lie in the groove smoothly without kinking. The problem was solved by making two modifications. The groove depth in the long arcs was made 0.011 inch deep which is 0.001 inch deeper than the wire is thick. When the wire bends around the small arcs it becomes slightly elliptical. To accommodate the change in the wire cross section, the groove was made 0.013 in deep in the arcs. To guide the wire into the grooves in the small joint arcs a special tool was constructed. At the center of each of the small arcs, a small hole was drilled. A step pin was inserted into this hole and used as a bending jig. The wire was bent around the jig which also served to guide it into the groove. This step pin is shown in Figure 2-17.
FIGURE 2-17. Heater Step Pin

The left hand side of the diagram shows a schematic of one of the small arcs which are used to construct the wire in the joints. Here the groove (in contrast to the previous diagrams which showed only the centerline path) in which the wire lies is shown. The right hand side shows the step pin which is used as bending jig and guide to insert the wire into the groove. The small point in the pin fits into a corresponding hole drilled at the center of each joint. The steps in the pin are used to constrain the wire while it is bent.

Once the wire is pressed into the groove, the top of the heater is filled with a thick layer of epoxy. The lip around the edge of the blank, shown in Figure 2-16 constrains the epoxy to make a pool about 0.1 inch deep. I used Epotek 301 epoxy which has a very low viscosity (about the same as water) and does not shrink appreciably as it hardens. The entire part is then put into a vacuum desiccator. The small bubbles of air which are trapped in the grooves expand and rise to the surface. The vacuum also removes most of the dissolved gases from the epoxy. The vacuum is then released, all remaining bubbles collapse, and the glue fills the grooves and seals the surface. The part is then left overnight to harden. Care must be taken to prevent the glue from getting on the underside of the blank.
After the epoxy has hardened, the part is put on a vacuum chuck and the lip and most of the epoxy is removed by fly-cutting the surface. This makes the top surface very flat and accurately parallel to the bottom surface. This explains why it is very important that no epoxy get on the bottom surface during the gluing process. The top surface is cut to within 0.003 inch of the embedded wires.

2.5.3 Bottom Plate Thermistors

A thermistor is used to measure the bottom plate temperature. The thermistor comes from the manufacturer (Fenwal electronics [6]) embedded in a 0.070 inch diameter glass housing. This makes it difficult to achieve good thermal contact with the sapphire. If it is simply pressed against the sapphire, the contact area would be too small to insure that they are in thermal equilibrium. To improve the thermal contact the thermistor head is first mounted in a copper capsule. This is achieved by machining a 0.125 inch diameter copper pill and drilling a 0.075 inch blind hole in the center. The pill is heated with a soldering iron and the hole is filled with solder. While the solder is molten, the glass head of the thermistor is inserted into the hole and the pill is allowed to cool. Although the solder does not wet the glass surface, the bulbous shape of the thermistor head insures that it is held firmly in place. Since the solder is in good mechanical contact with all sides of the thermistor head, the contact area is greatly
increased and the thermal contact is correspondingly improved. After the pill/thermistor assembly is cooled the excess solder is machined away and the head of the pill is polished. The assembly is then pressed against the sapphire by mounting it in a hole drilled in the excess plastic in the bottom plate heater as shown in Figure 2-18.

![Diagram of thermistor mounting](image)

**FIGURE 2-18. Mounting of Thermistor.**
This figure shows the mounting arrangement of the thermistor assembly in the plastic of the bottom plate heater. The crosshatched region shows the rubber gasket which spring loads the assembly. The polished end of the copper capsule extends 0.005 inches above the surface before it is pressed against the sapphire plate.
2.6 The Sample Flow System

2.6.1 The Flow Apparatus

The success of this experiment depends on generating, maintaining and measuring very slow flows. The flow system must be stable for days, should be as steady as possible, and shouldn't introduce contaminants. Common centrifugal laboratory pumps do not generate a very steady pressure head or volume flow rate. Peristaltic pumps are good at maintaining a constant flow rate, but create an undesirable pulsing action, which could make the analysis difficult. Another possibility is a reciprocating syringe pump, but like the peristaltic pump, there is an undesirable pressure glitch when the syringes change direction. Also, small pumps typically generate a much larger pressure head than is required. To generate a Reynolds number of around 1.0, one needs flow rates on the order of 1.0 ml/min. Any pump system must be throttled down to produce such slow flows. For these reasons, I decided not to use a simple pump arrangement to generate the pressure head. Instead I employ a system which uses a hydrostatic pressure to drive the flow. It uses a centrifugal pump to maintain the fluid level in two reservoirs. This is ideal for slow steady flows. By avoiding the direct use of a mechanical pump, vibrations and pulsations are greatly reduced. The pump system is shown in Figure 2-18. It consists of two reservoirs which are continuously filled by the action of the main pump (Cole-Palmer Model TE-5C-MD 1/8 hp
magnetic drive motor with plastic coated impeller [1]). The main pump drives a fluid loop with two filters. The first filter is a First Need™ water purification filter from General Ecology Inc.[2]. It consists of a combination of three filters: a 0.1 μm pore particulate filter, an activated carbon filter, and ion exchange resins. This filter was chosen because it is inexpensive and the integrated charcoal filter is good at removing hydrocarbons which slowly leach from the tubing and the cell sidewall material. It was discovered that some of the charcoal leaks out from the exit tube of this filter. This is harmless, but is inconvenient. To prevent this from passing into the flow system, a second filter was installed in the main loop.
This consists of a 10 cm diameter nitrocellulose particulate filter with 0.45 μm pores[3]. It is held by a stainless steel filter holder with teflon o-rings. It effectively traps the small amount of carbon which escapes from the carbon filter. Between the two filters are two pressure taps regulated with needle valves. These taps fill the two flow reservoirs. The reservoirs each consist of 1 liter Erlenmeyer flasks with a foamed silicone stopper in the neck. This stopper material was chosen because it was found that the standard black rubber laboratory stoppers leach contaminants into the fluid. These contaminants are presumably the sulfur compounds used to vulcanize the rubber, or the dye material which gives the stopper its black color. Additionally, the black stoppers were found to crack after time causing the flow system to leak. The silicone rubber stoppers seem to be chemically inert and have shown no signs of aging.

In the stopper are 4 tubes. The flask is partially filled with water, stoppered, and then inverted. The center hole has a 1/2 in polyethylene tube with a small funnel attached to the end. The funnel was machined down until it fit inside the neck of the flask. This forms the overflow tube. The other three holes are filled with 3/8 in tubing. These are the fill line, the exit line, and the atmospheric reference line. The fill line is connected to the pressure tap in the main pump loop. The exit line goes to the cell (in the upper reservoir) or returns from the cell (in the lower reservoir). The atmospheric reference line goes to the air space above the overflow tube.
Since each flask is separately referenced to atmosphere, the system is insensitive to changes in the local barometric pressure. The pressure head developed across the cell is set simply by the height difference between the overflow levels in the two flasks. The flow to the cell can be stopped with a valve on the exit line of the upper reservoir. This way, when the flow is valved off, the cell is subjected to a pressure head of the height of the column of water from the level of the lower reservoir. In normal operation, only the height of the upper reservoir is changed. In this way, the cell is subjected to the same pressure whenever the valve is closed regardless of the height of the upper reservoir. This attention to small pressures difference is necessary because I want to insure that the cell spacing remains constant. To prevent distorting the sapphires which form the cell plates, the bottom plate is clamped to the top plate only as tightly as is needed to seal the cell o-rings. Small changes in the cell pressure act on the large area of the plate, (79 cm\(^2\)) can change the plate spacing. With the pumping arrangement described, the change in the plate spacing which occurs when valves are closed is only two or three fringes as measured, \emph{in situ}, by the expanded laser beam. When I close a valve on the lower reservoir and leave the upper reservoir connected to the cell, the thickness change was as much as 30 fringes.

The entire system is plumbed with \emph{polyflow} brand polyethylene tubing with either plastic or stainless steel fittings. An early version of the flow
system was plumbed with tygon tubing, but this was found to have several problems. The first evidence of a problem was the discovery of traveling wave convection when there was no applied flow. This is the hallmark of binary fluid convection [4]. It appears that some sort of oil leaches out of the tygon tubing. One obvious indication of this effect is that after prolonged exposure to the water, the tubing becomes discolored and cloudy. Another problem is that the tygon is very soft. The polyflow is much more rigid. It was also found that pressure fluctuations of the bath water cooling loop were transmitted through the tygon to the sample fluid. The bath water would periodically squeeze the tubing. The convection rolls showed a high frequency jitter in addition to slow mean flow. When this line was re-plumbed, the jitter was eliminated.

Initially, I attempted to control the flow by leaving the pressure head constant and adjusting the flow by changing the flow impedance of the line with a teflon needle valve placed in the room. This didn’t work very well. It was found that the flow rate would fluctuate by as much as 10% from day to night. I believe this is due to the fact that the flow impedance depends on the viscosity of the fluid flowing through the valve. The valve was subjected to room temperature variations. Since the building services do a very poor job of regulating the room temperature, the valve would be subjected to at least 5°C variations from day to night (sometimes 10°C).
The viscosity of water varies by approximately 2%/°C. This variation would easily account for the observed variation in flow.

2.6.2 Measuring the Flow Rate

The original design measured the flow rate with a Riteflow® model flow meter from Manostar Inc. [5]. This meter consists of a conical glass tube with a ball inside the tube. The fluid flow is directed up the tube and the ball is held at a given level by the competition between the upward viscous drag imposed by the flow and the force of gravity on the ball. The height of the ball is a measure of the flow rate. The manual for this meter states that it has an overall accuracy of 2%. Perhaps it could achieve this accuracy under ideal conditions, but I found that the measured flow rate varied by as much as 20%. This was especially puzzling since the frequency of convection rolls was much more stable than this. The meter suffers from a combination of error sources. It is very dependent on the cleanliness of the ball and the tube. It was observed that the ball would sometimes stick at certain positions on the tube and could sometimes be dislodged by tapping or shaking. This was especially true for the slower flow rates. Moreover since this meter is based on the viscosity of the fluid, it is also susceptible to the variation of the viscosity of the water with room temperature. For these reasons, this flow meter was abandoned.
There are very few commercial flow meters which operate accurately at these slow flows. I decided to exploit the observation that convection in the absolutely unstable state is extremely regular. Moreover, the frequency of rolls passing any point is related to the mean flow rate. In fact, at constant $\varepsilon$, it is very nearly linear in the flow rate. The convection is measured with a video camera which is controlled by a quartz crystal oscillator, so one can measure the roll frequency extremely accurately and this method has very good resolution. The overall length of the time series is recorded using the real time clock in the PC, but the time base is the quartz crystal oscillator in the video camera. What is recorded is the number of frames which have elapsed since the beginning of the data run.

To measure the flow rate I settled on the following operational procedure. I set the temperature difference to a reference value $\Delta T = 3.2^\circ\text{C}$. This is close to the maximum $\Delta T$ I can use without the inlet region of the cell filling with convection rolls when there is no applied flow. I let the system stabilize for some time and then measured the frequency of rolls passing a point near the outlet. This is inserted into an empirically determined calibration to determine the mass flux of fluid passing through the cell.

2.6.3 Calibrating the Flow System

The calibration is performed by opening the flow system as illustrated by Figure 2-19, “Flow Calibration Setup”. I flow fluid from the upper reser-
voir though the cell and then into a syringe needle which empties into a small Erlenmeyer flask which is perched on top of a digital balance (Ohaus model 123). The syringe needle passes through a small hole in a sheet of parafilm which is stretched across the opening of the flask. This film prevents fluid from evaporating from the flask while the calibration is performed. The hole in the film was made large enough so that the syringe needle did not touch the edges of the hole. The bottom of the flask was filled with enough excess water that the hole in the syringe was submerged below its surface. This injects the fluid below the surface and prevents droplets from forming. It also insures that the pressure across the flow system is set by the difference between the level of the upper reservoir and the measuring flask and that there is no back pressure created by the surface tension of droplets. The water wets the syringe needle, forming a small meniscus. The surface tension of the water acting on the meniscus exerts a small force on the needle. However this force is present at both the start and end of the run. I measure only the difference in the weight of the system from the beginning to the end of each run so the surface tension force should cancel out.

As I increase the flow rate, the effective $\varepsilon$ decreases slightly since the linear stability boundary is a weak function of $Re$. I chose a fixed $\Delta T$, instead of a fixed $\varepsilon$, since to compute $\varepsilon$, one must know $\Delta T_c$ which itself is a function of the flow rate. Therefore I would have to iterate, setting the tem-
perature difference to the value which is approximately the desired $\varepsilon$, measure $Re$, and adjust the temperature to a more precise value.
2.7 Shadowgraph Apparatus

The shadowgraph apparatus is shown in Figure 2-20. It consists of a cylindrical tower of thin walled aluminum (1/16 inch) which contains a series of optical components. The tower mounts on top of the convection apparatus and has optical access to the convection cell through the glass window on the top of the cooling water bath.

The easiest way to understand the design and operation of the shadowgraph tower is to trace the path of light though the apparatus. The light source is a high intensity LED (H-3000-L “Super Bright” LED from Stanley Electric). It produces 3 candela of light at 660 nm which is coupled to a 50 μm graded index silica fiber from General Fiber Optics [7]. The fiber is approximately 1m long and is coiled into several loops. This coiling of the fiber serves to scramble the light so that an image of the LED chip is not projected from the other end of the fiber. The beam which emerges from the other end of the fiber has a roughly uniform intensity over its numerical aperture. The free end of the fiber is polished and serves as the light source. The silica fiber has a numerical aperture of 0.28 so the beam produced is a cone of light with an half angle of 16°. The free end of the fiber is mounted horizontally in an aluminum holder which allows one to translate and rotate the fiber angle over a small range. It is worth noting that most of this adjustability is unnecessary. In improved versions of the shadowgraph optics, the only adjustment that is
FIGURE 2-20. Assembly Diagram of Shadowgraph Tower
required is to translate the fiber end laterally. The cone of light emitted by the fiber impinges on a pellicle beam splitter mounted at 45° to the beam. Half of the light is projected down the tube. The other half strikes the side of the apparatus and is lost (a piece of black tape is placed on the side of the apparatus to help reduce stray light). The light then passes through a 50 cm, f/5 objective lens from Melles Griot. The fiber end is adjusted to be at the focal point of the objective lens so that the beam which emerges from the bottom of the tower is accurately parallel. The beam then passes through the glass window on the top of the apparatus, the cooling water bath, the top sapphire and into the convecting fluid. Here the lateral index of refraction gradients in the fluid act as weak lenses and refract the beam. A detailed analysis of this process is presented in *The Shadowgraph Method* on page 185. When comparing the results of this analysis to the experiment, it must be noted that the theoretical analysis is for an idealized single pass system, where the light passes through the sample once, whereas the instrument causes the light to pass through the sample twice; once on the way down and once on the way up.

Once the light has passed through the bottom sapphire it is reflected by the mirrored surface on the back face. If there were no convection present, the light would then retrace the exact same path up the tube. When it reaches the beam splitter, half of the light is reflected back towards the light fiber and re-focuses on its face. This allows one to accu-
rately adjust the position of the fiber end by moving it until the reflected spot is coincident with the returning beam. The other half of the light focuses in the air above the beam splitter assembly. An aluminum disk with a 1.0 mm aperture is mounted at this focus. This has the desirable property of blocking stray reflections which are created as the light passes through the various windows. It is, however, large enough to allow the beam (provided the convection is not too strong) to pass undisturbed. To aid in the elimination of spurious reflections, the glass window on the top of the convection apparatus is mounted at a slight angle to the horizontal to insure that its reflection is not coincident with the main beam and will therefore be blocked by the aperture. The reflections from the surfaces of the objective lens do not focus on the focal plane where the aperture is mounted, and are therefore rejected. The stray reflections from the two surfaces of the top sapphire and the top surface of the bottom sapphire do focus along with the main beam. Since the light source is an LED, its coherence length is short enough that the stray reflections do not interfere significantly with the main beam. They appear in the resulting shadowgraph image as a uniform background illumination which reduces the contrast sensitivity of the instrument.

The light then passes into the camera and lens system which images the resulting pattern onto the camera's CCD array. The camera and lens are separately mounted on movable platforms. Together they form a simple
magnifier which images a plane near the focal point onto the CCD element. The presence of the objective lens has the effect of re-mapping distances. It is straightforward to show that if one adjusts the camera and lens to image onto the CCD a plane a distance $x$ in front of the focal plane, one observes the shadowgraph pattern which would have appeared at a distance $z = \frac{f^2}{x} + L - f$, without the lens. This is shown in Figure 2-21 where $f$ is the focal length of the lens, and $L$ is the distance between the lens and the cell. The physical size of the image is given by the size of the cell times a magnification factor of $x/f$. One must adjust the magnification of the simple magnifier formed by the camera lens so that this image covers the entire CCD array. By imaging on a plane near
the focal spot, one observes the shadowgraph pattern which would appear at a great distance from the cell. If one images on the focal plane of the lens, it is equivalent to observing the shadowgraph pattern which would appear infinitely far from the cell. This is the Fraunhofer diffraction pattern of the shadowgraph pattern.

These concepts can be made clear if one considers a simple example. Consider imaging a plane 2.0 cm from the focal point of a 50 cm focal length lens. Let the lens-cell distance be 10 cm. The image that is observed is that which would appear a distance 1.21 m away from the cell if there were no objective lens present. If the cell were 10 cm in diameter, the image would be 0.4 cm in diameter. Since the CCD array in the camera is approximately 1.0 cm square, the camera and lens need to be adjusted to magnify the image by a factor of 2.5.

2.7.1 Problems with the Existing Design and Suggested Improvements.

The shadowgraph apparatus as described in the previous section works admirably, but the design could be improved. The biggest drawback with the design is the relative inaccessibility of the various components once the instrument has been assembled. The apparatus was mounted in a large thin walled aluminium tube since this structure is mechanically rigid and lightweight. This design decision, in my opinion, was a mistake. Once the apparatus has been assembled, it is impossible to adjust various
components such as the position and angle of the light source. It is also impossible to insert filters in the beam as an aid to calibration or to insert optical filters at the focal point of the objective lens (where the blocking aperture is located). The single biggest improvement would be to change the structure to an open rectangular frame with removable panels. In such a structure one can access the internal components while the apparatus is operating. Once everything has been adjusted satisfactorily, the side panels can be bolted on to provide light and dust shielding.

The choice of aluminum as a structural material was also somewhat unfortunate. The entire apparatus is exposed to the room temperature which is very poorly regulated. The room temperature can often vary by as much as 10°C (sometimes more!). The linear thermal expansion coefficient of aluminium is quite large $2.5 \times 10^{-5} / ^\circ C$ , so the apparatus can expand by around 0.02%. Since the apparatus is approximately 1 m long, the components may shift by as much as 0.2 mm. This shows up in the image in two ways. The image can shift from side to side and the overall magnification can drift slightly. A future apparatus should probably be constructed of a material with a lower thermal expansion coefficient. One possibility is to construct an optical rail from Invar rods with mounts for the optical components. The optical rail would be mounted in a external light shield. By separating the light shield from the mechanical portions
of the instrument, the optical components can be made much more accessible.

2.7.2 Image Processing

The video camera is connected to a PCeye® image grabber card from Chorus Data Systems [8]. This card is capable of digitizing an image which is 640 by 480 pixels. Each pixel is digitized to one of 256 intensity levels and is stored in one byte. For ease of data manipulation the image size is usually taken to be a 256x256 pixel square. The intensity noise is about 3 to 5 bit levels. The card is not a frame grabber; it can not acquire an image in a single frame. Instead it constructs an image by sampling lines from several succeeding frames, depending on the size of the image. Typically 6 frames (12 fields with interlacing) are required for a 256x256 image. This corresponds to 200 msec. This limits its use to images which change slowly compared to this time scale.

The goal of the shadowgraph apparatus is to accurately sample the intensity variations due to the shadowgraph focusing. There are several experimental details which must be addressed. Each sampled image has an unknown offset and gain so that the digitally sampled image \( I_{i,j} \) is related to the underlying intensity distribution \( I(x,y) \) by

\[
I_{i,j} = d_{i,j} + \alpha_{i,j} I(i \Delta x, j \Delta y),
\]

(2-13)

107
where the camera is assumed to respond linearly to intensity. This assumption ignores the averaging which occurs over the area of a pixel as well as the smearing of light from one pixel to another due to aberrations in the lenses. The dark signal is denoted \( d_{ij} \) which is assumed to vary from pixel to pixel. The per pixel gain is denoted \( \alpha_{ij} \). Both the dark signal and gain depend on the camera’s settings as well as the image grabber amplifier settings. They should be constant from image to image.

The shadowgraph pattern is proportional to the illumination intensity so one may write

\[
I(x, y) = I_0 S(x, y),
\]

(2-14)

where \( S(x, y) \) is the underlying shadowgraph focusing. This form assumes that the illumination intensity is spatially uniform. As will be shown in the geometrical optics analysis of the shadowgraph imaging (see Eq 4-23), there is a linear regime where the rays deviate only slightly between where they enter the cell and where they strike the image plane. In this limit we may model the angular variation of the light source intensity and the variations of the reflectively of the bottom plate replacing \( I_0 \rightarrow I_0(x, y) \) in Eq 2-14.

Within the linear regime, one may extract the pure shadowgraph signal with a combination of three images. The first image, denoted \( D_{ij} \), is a measurement of the per pixel black-point. It is created by extinguishing
the light source and taking an image (actually a large number of images are averaged to remove the image to image sampling noise). For the black-point image, \( I_0(x, y) = 0 \), so one has

\[ D_{ij} = d_{ij}. \]

(2-15)

The second image is a background image, denoted \( B_{ij} \). It is obtained by taking an image when the temperature difference across the cell is reduced below the convective threshold and the convection vanishes. For this image one has \( S(x, y) \Rightarrow 1 \) so the background image is

\[ B_{ij} = d_{ij} + \alpha_{ij} I_0(i\Delta x, j\Delta y). \]

(2-16)

The last image \( S_{ij} \), is obtained when the system is convecting. It is given by

\[ S_{ij} = d_{ij} + \alpha_{ij} I_0(i\Delta x, j\Delta y)S(i\Delta x, j\Delta y). \]

(2-17)

One may disentangle the sampled shadowgraph pattern from the gains and dark currents with the combination

\[ S(i\Delta x, j\Delta y) = \frac{S_{ij} - D_{ij}}{B_{ij} - D_{ij}}. \]

(2-18)

This computation is performed pixel-by-pixel on the digital images.

2.7.3 Removing Subpixel Image Shifts

The thermal expansion of the shadowgraph tower causes the background and sample images to drift relative to one another with time. These drifts
manifest themselves primarily as a translation of the origin of the two images. To a lesser extent, there is also a change in the scale as well as a rotation and skew. The dark image is presumed to be independent of the room temperature since it is a characteristic of camera and digitizing board, although both of these components could be affected by the room temperature. A simple test with a heat gun used to heat both the computer board and the camera didn’t show any obvious temperature dependence.

When the two images are slightly misaligned some features on the background image are not completely divided out. This problem is particularly acute when images taken very near the convective threshold are divided as is demonstrated in Figure 2-22. The subpixel shifts are performed by a bilinear interpolation of neighboring pixels

\[
I(x, y) = [ (1 - \Delta x)(1 - \Delta y)I_{i,j} + \Delta x(1 - \Delta y)I_{i+1,j} \\
\Delta x(1 - \Delta y)I_{i,j+1} + \Delta x\Delta yI_{i+1,j+1} ]/4
\]

\[
i = \text{Floor}(x) \\
j = \text{Floor}(y) \\
\Delta x = x - i \\
\Delta y = y - j.
\]

(2-19)

In this formula, \text{Floor}(x) is the largest integer smaller than \( x \). The shifts required to align the images have always been less than one pixel in both \( x \) and \( y \). When images are misaligned, the divided image shows a characteristic three dimensional highlight on the edges of the cell. This is an artifact
FIGURE 2-22. Effect of Sub-Pixel Shift

This figure demonstrates the effect of shifting a shadowgraph image with respect to its corresponding background image. All images are taken with the flow turned off. Image a was taken with a temperature difference of 2.300 °C across the cell, which is below the convective onset. Image b was taken at 2.500 °C which is very close to the convective onset for this cell. No rolls can be seen in the unprocessed image. Image c is the division of b by a without shifting. The highlights on the edges of the cell indicate that the center of the two images are no longer coincident. Image d is the same image division after the background has been shifted downward and to the left by (0.2, 0.2) pixels. Notice that the highlights are reduced and the convection rolls are more clearly visible.
caused by a dark highlight on one side of the cell formed by the division of a dark wall pixel by a bright cell pixel and a bright highlight on the other side of the cell by the converse division. Usually this procedure is applied as an *ad hoc* cosmetic correction, although a rigorous criterion could presumably be found based on the highlights observed in the division of two background images.


3.1 Introduction

The central goal of the experiment is to measure the system in the convectively unstable regime and to determine the strength of the noise that drives the convection. To this end, the convective and absolute stability boundaries are measured as a function of the applied Reynolds number. Once the parameter space has been mapped out, the system is put into the convectively unstable state and the amplitude as a function of position is measured for several flow rates and temperature differences. The measurements are compared to a numerical simulation of the complex Ginzburg-Landau equation (CGL) with a stochastic forcing term. The noise strength is adjusted until the simulation and the experiment agree. The amplitude of the noise term in the CGL is a measurement of the experimental noise strength. This is compared to the noise strength one
would expect from thermal fluctuations. There are four data sets, each of which was taken with a different flow rate. I will describe the analysis for one of these data sets and then will present the corresponding results from the others.

3.2 Measuring the Critical Temperature Difference

The critical temperature difference at which convection first becomes unstable is a weak function of the applied flow rate. The stress parameter \( \varepsilon - \varepsilon^T_{\text{conv}} \), which appears in the theory, is the dimensionless temperature difference from the onset of convection at the applied flow rate. Therefore it is essential to measure the critical temperature difference as a function of the flow rate.

When the cell is first assembled, it is characterized without the applied through flow to determine the critical temperature difference \( \Delta T_c(\text{Re} = 0) \). The temperature difference across the cell is stepped through the convective onset. The power dissipated in the bottom plate is measured at each point and an image of the cell is taken. One may extract \( \Delta T_c(\text{Re} = 0) \) in two ways. The most straightforward is to use the Nusselt number \( N \), which is a dimensionless ratio of the amount of heat transported in the presence of convection to that transported by conduction alone, \( N = \lambda_{\text{conv}}/\lambda_{\text{cond}} \). Near the convective onset, the quantity \( N - 1 \) is
directly proportional to the reduced temperature difference,
\[ \varepsilon = \Delta T / \Delta T_c - 1 \]. Extrapolating \( N - 1 \) to zero allows one to determine \( \Delta T_c \). The second method is to measure the convection amplitude directly from the shadowgraph images. One may similarly extrapolate the square of the amplitude to zero to find \( \Delta T_c \).

3.2.1 Extrapolating the Nusselt Number

The heat current data measures the total power dissipated in the bottom plate heater as a function of the applied temperature difference across the cell. To compute the Nusselt number, one must determine how much of the heat flows through the fluid in the cell and how much goes through the sidewall and air. I employ a simple model of the thermal properties of the cell which is shown schematically in Figure 3-1. The cell and sidewall

![Figure 3-1. Schematic Thermal Model of the Cell](image)

The cell and sidewall are considered to be thermally parallel. The combination is in series with two sapphire plates. Note that the sidewall conductivity really includes all stray conductance which is not through the cell. The temperature difference are shown on the left and the slab thicknesses are shown on the right.

are considered to be thermally parallel and in series with the two sapphire
plates. The temperature difference is measured across the entire system. The temperature differences across the system, the sapphire, and the cell are denoted \( \Delta T \), \( \Delta T_s \), and \( \Delta T_{\text{cell}} \) respectively. The thickness of the sapphire and the cell are \( d_s \) and \( d \). One may correct for the temperature drop across the sapphires by writing \( \Delta T = 2\Delta T_s + \Delta T_{\text{cell}} \). One may use the known conductivity of sapphire to write \( \Delta T_s = Q d_s / (\lambda_s A_s) \) where \( Q \) is the heat produced by the bottom plate heater (all of which is assumed to flow through the sapphires), \( \lambda_s \) is the conductivity of sapphire, and \( A_s \) is the area of the sapphire through which the heat flows. The area \( A_s \) is approximated by area covered by the sidewall and the cell since little heat flows through the air outside the cell. Since the correction for the temperature difference across the sapphire is small (less than 10%), little error is introduced into the calculation by this approximation. Therefore one may write the temperature drop across the cell as

\[
\Delta T_{\text{cell}} = \Delta T - 2 \frac{Q}{\lambda_s A_s} d_s. \tag{3-1}
\]

Next one must compute the heat which flows through the cell and how much flows through the sidewall. Since they are in parallel, their conductances add and one has \( C = C_{\text{wall}} + C_{\text{cell}} \) where \( C = Q / \Delta T \), \( C_{\text{wall}} = Q_{\text{wall}} / \Delta T_{\text{cell}} \), and \( C_{\text{cell}} = Q_{\text{cell}} / \Delta T_{\text{cell}} \). The cell conductance may further be written as the sum of two parts: that due to conduction and that due to convection \( C_{\text{cell}} = C_{\text{cond}} + C_{\text{conv}} \). If the temperature dif-
ference across the cell is below the critical temperature difference \( C_{\text{conv}} \) is zero and \( C_{\text{cond}} \) is known from the conductivity of water and the cell geometry \( C_{\text{cond}} = \lambda_{\text{water}} A/d \), where \( A \) is the area of the wide portion of the channel and \( d \) its thickness. Therefore one has

\[
C = C_{\text{wall}} + C_{\text{cond}} + C_{\text{conv}},
\]

in general and

\[
C_{\text{below}} = C_{\text{wall}} + C_{\text{cond}},
\]

below the transition. One may eliminate the wall conduction by forming the difference

\[
C - C_{\text{below}} = C_{\text{cond}}.
\]

I perform a fit of all of the points below the transition to get an accurate measurement of \( C_{\text{below}} \). Now one may write the Nusselt number as

\[
N = \frac{\lambda_{\text{conv}}}{\lambda_{\text{cond}}} = \frac{(C - C_{\text{below}}) d}{\lambda_{\text{water}}} A.
\]

Typical Nusselt number data is shown in Figure 3-2. Once one has computed the Nusselt number, one can extract \( \Delta T_c \) as the extrapolation of the upper branch of the Nusselt number curve. A more precise method is to fit the shape of the entire Nusselt number curve. Since the through flow is turned off \( v_t = c_0 = c_2 = 0 \), and one is left with the Ginzburg-Landau equation [1, 2].

\[
\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g|A|^2 A.
\]
FIGURE 3-2. Nusselt Number Measurement without Through Flow

The circles are data points taken with increasing temperature differences in 50 mK steps. The squares are data points with decreasing temperature differences. At each point the cell is allowed 3000 seconds to equilibrate. Each measurement was an average of 300 samples taken 3 seconds apart so that fluctuations of the temperature servos are reduced. Once the maximum temperature difference was reached, another data run was taken stepping down. Since the downward data reproduce the upward data, one concludes that there was little or no drift during the data run. The line shows the results of a fit to the real GL equation.

The coefficients of this equation were computed for an infinite system with nonslip boundary conditions on the top and bottom surfaces by Cross [3]. If one assumes that the amplitude is constant along the channel then \( \partial_x^2 A = 0 \), and Eq 3-6 reduces to the Landau equation in which \( A \) may be taken as purely real.
\[ \tau_0 \partial_t A = \varepsilon A - gA^3. \quad (3-7) \]

This Landau equation has a pitchfork bifurcation at \( \varepsilon = 0 \) where the steady-state solution (\( \partial_t A = 0 \)) exchanges stability from the trivial \( A = 0 \) for \( \varepsilon \leq 0 \) to the finite solution \( A = \pm \sqrt{\varepsilon / g} \) for \( \varepsilon > 0 \). The two solutions for \( \varepsilon > 0 \) represent the degeneracy which exists between rolls which overturn clockwise (at the some point in the cell) and those which turn counter-clockwise. If \( \varepsilon \) is not perfectly uniform, or if one includes the effects of the ends of the cell, the amplitude varies slightly along the channel and the sharpness of the bifurcation is spoiled. Moreover, thermal boundary conditions near at the ends of the cell will either favor upflow or downflow. This breaks the symmetry of the positive and negative solutions. Since the Nusselt number is a global measurement of heat flow, it always shows this rounding to some extent. If the variation of \( \varepsilon \) is not known explicitly one may model the rounding by adding a phenomenological forcing term to the right hand side of Eq 3-7. The size of this forcing term gives a measure of the perfection of the cell geometry and \( \varepsilon \) uniformity. The effect of this forcing term is shown schematically in Figure 3-3. The two symmetrical branches of the amplitude, which carry the same heat current, are split. Since the measurement is performed by stepping slowly through the transition, presumably the system remains on the upper branch. For steady state solutions, Eq 3-6 reduces to
FIGURE 3-3. Effect of Adding a Field to the Landau Equation

The figure on the upper right shows the pitchfork bifurcation of the Landau equation. The symmetry between the positive and negative branches is broken by the addition of a field as shown in the plot on the upper right. The dashed lines show the unstable solutions. The plot on the lower right shows the Nusselt number which results from the pure Landau equation. The lower right image shows the Nusselt number from the upper branch of the Landau equation with a field.

\[ 0 = (\varepsilon - gA^2)A + f. \]  \hspace{1cm} (3-8)

One can now solve this cubic polynomial to find \( A(\varepsilon) \) which allows one to compute the Nusselt number via the relation

\[ N = 1 + A(\varepsilon)^2/(1 + \varepsilon). \]  \hspace{1cm} (3-9)
This relation can be fit to the Nusselt number data recalling
\( \varepsilon = \Delta T / \Delta T_c - 1 \). One lets \( \Delta T_c \), \( g \), and \( f \) be adjusted as fit parameters.
The results of this fit are shown in Figure 3-2 and the coefficients are summarized in Table 3-1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta T_c )</td>
<td>2.523 ± 0.002 °C</td>
<td>2.681 °C</td>
</tr>
<tr>
<td>( g )</td>
<td>1.47 ± 0.01</td>
<td>0.699</td>
</tr>
<tr>
<td>( f )</td>
<td>( (4.59 \pm 5.4) \times 10^{-5} )</td>
<td>0</td>
</tr>
</tbody>
</table>

It was discovered that the critical temperature difference was drifting downwards monotonically with time. This is probably due to the sidewall material absorbing water and swelling. The critical temperature difference depends on the cube of the cell thickness so it changes dramatically when the cell thickness changes only slightly. The observed drift is consistent with a change of less than 0.002 inches out of an initial thickness of 0.1274 inches. To characterize the onset drift, periodic measurements of the onset were performed, starting when the cell was originally assembled. The critical temperature difference is plotted against time in Figure 3-4.

In the theory all of the boundaries are defined relative to the onset for \( \text{Re}=0 \), so it is important to use the critical temperature difference when the measurements were made. Therefore I have fitted a phenomenological
interpolating function to the measurements of the critical temperature difference. The function used was

\[ \Delta T_c(t) = A e^{Bt} + C, \tag{3-10} \]

where \( A, B, \) and \( C \) are fit parameters. There is no theoretical justification for this fit function other than that it smoothly interpolates the data and it is reasonable to expect that the swelling would obey a relaxation process. The data and fit are shown in Figure 3-4 and the fit parameters are sum-

- \begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure3-4.png}
  \caption{Drift of Critical Temperature Difference with Time}
  \end{figure}

This figure shows the change in the critical temperature difference as the experimental data were taken. The onset was remeasured periodically. This abscissa is the number of days elapsed from the cell assembly date: Oct 3, 1994.

marized in Table 3-2.
Table 3-2. Phenomenological Fit Parameters to Critical Temperature Difference

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.1157 °C</td>
</tr>
<tr>
<td>B</td>
<td>$-7.778 \times 10^{-3}$ day$^{-1}$</td>
</tr>
<tr>
<td>C</td>
<td>2.518 °C</td>
</tr>
</tbody>
</table>

3.3 Measuring the Amplitude as a Function of Position

In this section I describe how the amplitude is measured as a function of distance. The procedure is somewhat involved. Much of the complexity derives from the nontrivial cell geometry. The procedure has several steps. First one must locate the active region of the cell in the image and find the center line. The cell is then numerically mapped, or to borrow a term from the movie industry, *morphed* into a rectangle$^1$. I use a conformal mapping which locally preserves angles and distance along the center line. This is important since the distance is an important parameter in the theory and it would be incorrect to have the image processing change it. This procedure also allows one to correct for optical distortions in the shadowgraph optics and a non-unity pixel aspect ratio. The aspect ratio is defined as the ratio of the $x$ and $y$ distances in the cell which are measured by each CCD pixel. It is not necessarily the same as the physical dimensions of the CCD element since it includes any distortions introduced by the align-

---

1. See a good technical reference like the movies *Willow* and *Terminator 2* which make extensive use of this technique.
ment of the optics. Once the cell has been unrolled, I average pixels across the channel to improve the signal to noise ratio.

3.3.1 Finding the Center Line of the Spiral

After the shadowgraph apparatus has been adjusted, one must find the center of the channel. The procedure begins by finding the pixels which bound the image of the channel. One could simply threshold the image and use all of the pixels which are brighter than a given value as parts of the channel. This has a couple of problems. The shadowgraph apparatus does not image the surfaces of the cell onto the video camera. Therefore the edges in the images are not sharp but instead are blurred out over several pixels. Also the nonuniform illumination of the image and nonuniform bottom plate reflectivity means that one rejects more pixels in regions of the image that are dark than one does where it is light. This distorts the measurement of the center of the channel. Instead I convolve the image with a smoothing edge enhancement filter to locally differentiate the image and find the areas of high contrast. This has the effect of finding the transitions of light to dark (which mark the edges of the cell) but it rejects high frequency noise in the image which is due to camera noise and marks on the bottom plate. It also is less sensitive to local variations in the illumination. To understand the effect of this filter, I will briefly
review a little of the theory of digital signal processing. The filter can be written in the general form

\[ \bar{I}(x) = \int_{-\infty}^{\infty} f(x - x')I(x')dx', \quad (3-11) \]

where \( f(x) \) is the filtering function and \( I(x) \) is the measured signal. If one chooses the filter to be a delta function, \( f(x) = \delta(x) \), then Eq 3-11 becomes an identity, \( \bar{I}(x) = I(x) \). If one approximates the delta function by a function which is peaked about \( x = 0 \), such as a gaussian, then \( \bar{I}(x) \) becomes a smoothed version of \( I(x) \). It is smoothed, since each point in \( \bar{I}(x) \) is a weighted average of neighboring points within approximately \( x \pm w \), where \( w \) is the width of the filter. The wider the filter, the more averaging is performed, and the smoother is \( \bar{I}(x) \). This is most easily understood if one takes the Fourier transform of Eq 3-11. One has

\[ \bar{I}(k) = f(k)I(k). \quad (3-12) \]

If \( f(x) \) has a width of \( w \), then \( f(k) \) has a width of \( 1/w \). Spatial frequencies larger than this are suppressed. The filter I used is an edge enhancement filter. It differentiates the signal. It is based on the observation that one can construct the derivative of a signal by choosing the filter to be the derivative of the delta function

\[ \frac{d}{dx} I(x) = \int_{-\infty}^{\infty} \frac{d}{dx} \delta'(x - x')I(x')dx'. \quad (3-13) \]
One may again approximate the delta function by a peaked function to obtain a smoothed version of the derivative. This filter has the effect of enhancing edges and smoothing high k features with a length scale set by w. In Fourier space Eq 3-13 becomes

$$\tilde{I} = kf(k)\tilde{I}(k),$$  \hspace{1cm} (3-14)

To apply this to the image data, I choose $f(x)$ to be a gaussian. Once the edges of the cell have been enhanced, the filtered image is thresholded so that only the edges are shown. The results of this operation are shown in Figure 3-5.

![Figure 3-5. Edge Enhancement of Background Image](image)

The left image is the background image which is taken below the convective onset. The center image is the edge enhanced version image. The right image is a thresholded version of the center image.

The next step is to identify the boundary edges by starting from some point near the center of the channel and working both outwards and inwards until the boundaries of the cell are identified. The $(x, y)$ coordi-
nates of each boundary edge are recorded. Since comparison to theory will be made using the distance along the centerline, one must compute the centerline by averaging the radii of the inner and outer edges. This requires an accurate determination of the center of the cell. Unfortunately this can only roughly be determined by eye. The problem is particularly obvious if one plots the centerline as a function of angle as shown in Figure 3.9. The centerline is constructed so that it should be a linear

![Graph](image)

**FIGURE 3-6. Radius of Centerline as a Function of Angle**

This figure shows the inner and outer boundaries of the cell. The average of these two boundaries gives the centerline. It is clear that the measured centerline radius is not a linear function of the angle which indicates a poor determination of the origin of the cell (which was set by eye).
function of the angle. If one has an incorrect determination of the origin, the measurement of the centerline will wobble. Another problem which can cause a similar distortion is if the camera pixels are not square. This data can be used to measure and correct for these distortions. In particular I use a nonlinear least squares routine to minimize

$$
\chi^2 = \sum_{i=1}^{N} [r_i - (r_0 + k\theta_i)]^2
$$

$$
\theta_i = \tan^{-1} \left( \frac{y_i - y_0}{x_i - x_0} \right)
$$

$$
r_i = \sqrt{\alpha^2(y_i - y_0)^2 + (x_i - x_0)^2}.
$$

Here, the coordinates of the centerline are \((x_i, y_i)\) and the fit parameters are the coordinates of the origin \((x_0, y_0)\), the pixel aspect ratio \(\alpha\), the intercept \(r_0\), and the pitch of the spiral \(k\). The two useful results of this fit are that the pixel aspect ratio is determined and that the pitch of the spiral can be compared to the known spiral pitch which allows one to convert pixels to physical distances. The results of this fit are shown in Figure 3-7. and the relevant fit parameters are summarized in Table 3-3. The origin can be determined with an accuracy of less than one pixel. The errors however, are the statistical error of the fit parameters and are overly optimistic. They ignore the systematic errors associated with optical distortions and cell imperfections. The reproducibility of the results from image to image, vary by about 0.5 pixels. The pixel aspect ratio is nearly,
FIGURE 3-7. Radius of Centerline as a Function of Angle (II)

This graph shows the results of the fit which finds the correct origin and pixel aspect ratio. The widening of the channel marks the beginning of the active region of the cell.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$125.55 \pm 0.02 \text{ pixel}$</td>
</tr>
<tr>
<td>$y_0$</td>
<td>$116.64 \pm 0.02 \text{ pixel}$</td>
</tr>
<tr>
<td>$r_0$</td>
<td>$19.42 \pm 0.03 \text{ pixel}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$4.644 \pm 0.003 \text{ pixel/radian}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$0.9671 \pm 0.0004$</td>
</tr>
</tbody>
</table>

but not exactly, one. Therefore the distances will all be quoted in terms of the x width of a pixel. The pitch machined into the sidewall channel is
0.15160 cm/radian so one may use the measured spiral pitch to compute a conversion from pixel widths to cm as 30.632 pixel/cm.

3.3.2 Unrolling the Spiral

In this section I will explain how the spiral channel is morphed into a rectangular one. First one must sample the spiral in uniform increments of the arc length along the centerline. Write the equation of the spiral as

\[ r(\theta) = k\theta \]

where I have chosen the origin of \( \theta \) so that \( r_0 = 0 \). One may now compute the arc length \( s \), as

\[
\begin{align*}
  ds^2 &= dr^2 + r^2 d\theta^2 \\
  ds &= d\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + 1} \\
  s &= k \int_0^\theta d\theta \sqrt{1 + \theta^2} \\
  &= \frac{k}{2} \left( \theta \sqrt{1 + \theta^2} + \sin^{-1}\theta \right). 
\end{align*}
\]

This implicitly defines the function \( \theta(s) \) which allows one to easily sample the spiral in uniform increments of the arc length. Now I want to sample the points perpendicular to the curve. This is most easily understood if one considers a general polar curve \( r(\theta) = r(\theta)\hat{r}(\theta) \). The unnormalized tangent vector is given by

\[
  r'(\theta) = \frac{d}{d\theta} [r(\theta)] = r'(\theta) \hat{r}(\theta) + r(\theta) \hat{\theta}(\theta) .
\]
where \( \mathbf{f}(\theta) \) and \( \mathbf{\hat{\Theta}}(\theta) \) are unit vectors in polar coordinates. The unnormalized outward pointing perpendicular vector is given by

\[
\mathbf{r}_\perp(\theta) = r(\theta)\hat{\mathbf{r}}(\theta) - r'(\theta)\mathbf{\hat{\Theta}}(\theta).
\]

(3-18)

Now one may resample the image into a rectangular array \( R_{ij} \) by letting the \( i \) index run along the \( s \) direction and the \( j \) index run along the local perpendicular vector. Therefore one has

\[
R_{ij} = I\left(r\hat{\mathbf{r}} + \mathbf{r}_\perp \left(j - \frac{nr_r}{2}\right)\Delta r\right),
\]

(3-19)

where \( r, \hat{\mathbf{r}}, \) and \( \mathbf{r}_\perp \) are functions of \( \Theta(i\Delta s) \). The spatial increments of radius and arc length are \( \Delta r \) and \( \Delta s \) which are chosen to be one pixel. The radial index \( i \) runs from 1 to \( n_r \) and the length index runs from 1 to \( n_s \). The function \( I(x) \) is an interpolating function which is derived from the image samples \( I_{ij} \) by the filter

\[
I(x) = \frac{\sum_{i,j} I_{ij}f(|x - (x_0 + ix + jy)|)}{\sum_{i,j}f(|x - (x_0 + ix + jy)|)}.
\]

(3-20)

The function \( f(x) = \exp(-x^2/2\sigma_x^2) \) is chosen to be a gaussian filter with a width \( \sigma_x \) of one pixel. The results of the unrolling algorithm are shown in Figure 3-8. A fit of the shadowgraph signal in the interior 2/3 of the active region of the cell to a sinusoid gives a measurement of the wavevector of the pattern. One finds \( k_{\exp} = 2.829 \). The theoretical prediction for the critical wavevector is \( k_{\text{th}}(\text{Re} = 0) = 3.116 \). The discrep-
FIGURE 3-8. Unrolling the Spiral

This figure shows the shadowgraph of a cell without an applied flow. The upper panel shows the divided image. The middle shows the unrolled spiral and the bottom shows the intensity amplitude averaged across the channel. I have overlayed an image of the unrolled spiral as an aid to identifying peaks. This shadowgraph image was taken at $\varepsilon = 0.0059$.
ancy between the measurement and the prediction is likely due to the
curvature of the channel. Recall that I am measuring the amplitude along
the centerline by averaging across the channel. The rolls on the outside of
the channel are therefore stretched and the rolls on the inside are com-
pressed. The wavevector on the outside of the channel is lower than the
measured value and on the inside of the channel it is correspondingly
higher. One can estimate the range of wavevectors from purely geometric
considerations. The actual wavevector in various different radii in the
channel is given by \( k(r) = k_{\exp} r_{\text{mid}} / r \) where \( r_{\text{mid}} \) is the radius of the
middle of the channel where the wavevector is measured. At the cell inlet
one has, \( k_{\text{outer}} = 2.516 \) and \( k_{\text{inner}} = 3.174 \). The range spans the theoreti-
cal value.

To convert the unrolled spiral to a measure of the one dimensional amphi-
tude, the pixels across the inner two thirds of the channel are averaged.
This has the effect of reducing the noise. This procedure works well for
images of the cell when there is no applied flow and one may take as
much time as is required for the computation of the amplitude. When the
flow is applied, the roll images move. The shortest period was about 15
sec. Ideally one would like to have at least 10 samples per second, so I set-
tled on trying to achieve a sample rate of at least 1Hz. It was impossible to
take and store images at this rate especially when time series of 1024
images were desired. The storage requirements were also prohibitive.
Each data run would require 64Mbytes of storage. To meet the goal I had
to reduce the amount of data stored and limit the processing which is
done during each image. Consequently I reduced the number of locations
sampled from every pixel along the center line (661 samples) to one sam-
ple every 6 pixels (111 samples). Additionally the two steps described
above, unrolling and averaging, were combined into a single step. At each
position along the centerline I computed the filtered value by averaging
an angular wedge of pixels with the (unnormalized) filter

$$f(\Delta r, \Delta \theta) = R(\Delta r) \Theta(\Delta \theta)$$

$$R(r) = (1 + \tanh[(r + r_w)/\sigma_r])(1 + \tanh[(r_w - r)/\sigma_r])$$

$$\Theta(\theta) = e^{-\theta^2/(2\sigma^2)}$$

(3-21)

where $\Delta r$ and $\Delta \theta$ are the radial and angular distances from a points on
the spiral. Both are measured from the local center of curvature of the
centerline; not the center of the spiral. This insures that the pixels are
properly averaged across the channel. The radial filter deserves some com-
ment. This form is chosen since it has a flat central region which weights
the pixels in the center of the channel uniformly and it goes to zero
smoothly. It has two parameters, $r_w$ and $\sigma_r$, that set the radial width of
the filter and the sharpness of the radial fall off; they are illustrated in
Figure 3-9. The radial filter width is chosen to be $2/3$ of the channel
width and $\sigma_r$ is chosen to be one pixel. The angular function is a simple
gaussian. The angular width $\sigma_\theta$ is chosen to be one pixel on the center of
FIGURE 3-9. Radial Filter

This figure shows the radial filter used to average pixels across the channel. It has two adjustable parameters, \( r_w \) and \( \sigma_r \), which control the width and fall off of the filter. This filter is centered on the centerline of the channel.

The sampled pixels are shown in Figure 3-10. The controlling computer is fast enough to take an image, compute the filter and store the results in less than the design goal of one second. The filtered data for each pixel is represented in a long integer (4 bytes) so that the data for one image is reduced from the raw data of 65536 bytes to 440 bytes. Each data run consists of a time series of 1024 images. The total storage for each data run is therefore 450560 bytes which when compressed using a standard compression routine (Lempel-Ziv [4]), is reduced by about 30% to approximately 350 kilobytes. The overall data reduction is approximately 99.5% (from 64Mb to 350 kb). Once the data
FIGURE 3-10. Pixels in Filter

This image shows the pixels which are used to sample the intensity along the spiral. The gray scale is proportional to the weight associated with that pixel. There are 111 sample locations spaced 6 pixels apart.

has been filtered, it can be image divided using the same algorithms used for the unfiltered images. See Image Processing on page 107.

3.4 Measuring the Critical Temperature Difference as a Function of Reynolds Number

Thus far I have only described measurements performed on the system without the applied flow. In this section I will remove this restriction. One may no longer easily use the heat current measurements to find the
critical temperature difference. There are two reasons. As one increases the temperature difference across the cell, the rolls first appear near the outlet. Further increasing the temperature difference causes two things to happen. The amplitude of the rolls increases and the fraction of the cell filled with convection increases. This fraction depends on the length of the cell. Moreover since the convection near the onset is presumably noise driven, the fraction of the cell filled fluctuates in time. One can average the amplitude over time to get an ensemble average, but the net result is that the Nusselt number no longer shows a sharp transition at $\Delta T_c$. The problem is illustrated in Figure 3-11. The two data runs were taken

![Figure 3-11. Nusselt Number with Applied Flow](image-url)

This figure shows the measured Nusselt number with an applied through flow of $Re=0.850$ and the equivalent data for $Re=0$. The Nusselt number data is rounded since only a fraction of the cell is filled.
within a couple of weeks of each other so the drift of the critical temperature difference due to the sidewall swelling, is not important. The heat current is rounded and does not show a sharp transition. Moreover, the critical temperature difference that one infers from this data is an overestimate of the true onset. It can only detect convection when it has grown to a measurable amplitude before fluid has exited the cell. The length over which this occurs diverges as one approaches the onset. If this healing length is much longer than the cell, one will not observe any extra heat transport, even though the system is above the convective threshold.

Fortunately, there is a measurement technique which is free of these deficiencies. The convective onset is defined at the temperature difference where a perturbation at the critical wavevector is neutrally stable. That is, it neither grows nor decays as it propagates through the cell. Above the onset, there is a range of wave vectors which grow. This can be directly measured by applying a perturbation at the inlet of the channel and measuring the amplitude as a function of distance as described above. If the perturbation grows as it propagates, the system is above the neutral curve. The spatial growth rate measures how far above onset the system is. One may extrapolate these measurements to find the onset. This method has the advantage of being independent of the channel length and therefore is presumably measuring a property which would persist in an infinitely long system.
To measure the spatial growth rate one could simply apply a pulse to the perturbation heater and measure the resulting convection. A simple example of this procedure is shown in Figure 3-12 which shows the response of the system to a heat pulse generated by the perturbation heater. In this image the spiral has been digitally unrolled so that the distance along the centerline is on the x axis. Time is along the vertical axis, and the gray scale at each point is proportional to the shadowgraph signal. This image shows a space time plot of the evolution of the pulse. It is observed as a white patch near the lower left. Since the heat pulse is dissipated in the sidewall, it primarily generates a small roll parallel to the channel. This roll is not stable and it begins to decay. However, in the process a small amplitude of the critical mode is excited and it is seen to grow and expand as it travels towards the upper right. It is clear that the perturbation is being advected faster than it spreads so that it exits the cell and the amplitude eventually decays everywhere in the cell. The system is, by definition, in the convectively unstable state.

This procedure, while unequivocal, is difficult to apply when the system is very near the convective onset since the spatial growth rates are small. One may achieve much better accuracy by applying a continuous harmonic perturbation. In this method a sinusoidal voltage modulation

\[ V(t) = V_0 \cos(\omega t) \]

is applied to the perturbation heater. The temperature perturbation induced couples to the convection in some complicated
FIGURE 3-12. Space Time Plot of Response to a Perturbation Heater Pulse.

This figure shows the response of the system to a heat pulse generated by the perturbation heater. The gray scale is proportional to the shadowgraph signal. The spiral has been digitally unrolled and the horizontal axis measures the distance along the centerline. The vertical axis measures time. The pulse, which is seen as a white spot on the lower left, grows and spreads as it travels downstream and exits the cell at the outlet. Since the amplitude at every point in the lab frame eventually decays to zero, this system is by definition in the convectively unstable state.
way but so long as it is small, it should be proportional to the power dissipated in the heater. The temperature perturbation is proportional to the square of the voltage and has twice the applied frequency as well as a DC component, \( T(t) = T_0[1 + \cos(2\omega t)] \). It will be shown that the DC component is extremely strongly damped so that the system is essentially subjected to a pure sinusoidal temperature modulation at \( 2\omega \). The amplitude is proportional to the local temperature, so in the simulations of the system, the thermal perturbation may be treated as a local perturbation of the amplitude, although of an unknown strength. In all of that which follows, the frequency which will be reported will be \( 2\omega \), the frequency of the temperature perturbation, which is twice the frequency of the voltage drive.

In the driven experiments, one wishes to extract the amplitude which is the response to the drive. In general, the measured convection will be a combination of the noise driven convection and the perturbation driven convection. When the measured convective amplitude is small compared to saturation, the response to the perturbation is linear; the driven convection will be at the drive frequency. Conversely, one expects that the noise driven convection will occur at a superposition of unstable frequencies. One may greatly simplify the data analysis and increase the sensitivity of the measurement by bandwidth limiting the data collection to the drive frequency. This is easily achieved by computing a discrete Fourier
transform of the time series and extracting the power in the bin which corresponds to the drive frequency. Usually when one estimates signal power in this way, one must window the data in real space to prevent spurious high frequency components from being aliased into the data due to the finite sampling interval [5]. (See APPENDIX F The Effect of Windowing on DFT Power Spectra on page 275.) However, one can avoid this problem entirely by choosing the drive frequency to be a multiple of the fundamental sampling frequency \( \pi/(N \Delta t) \), where \( N \) is the number of samples and \( \Delta t \) is the sampling interval. When the drive frequency is chosen in this way, there is an integral number of drive periods in the sample and all of the power at the drive frequency appears in a single Fourier bin. The measurement therefore consists of taking 1024 images at one second intervals. The amplitude at the 111 locations along the channel is extracted and stored. One then analyses the data by computing a radix-2 fast Fourier transform of the time series at each location and extracting the power in the Fourier bin which corresponds to the drive frequency. An example of this is shown in Figure 3-13. The square root of the power is a measure of the time averaged amplitude at that point. One then performs the image division at each location. One may use a separate background image which is taken when there is no convection present, but one can alternatively use the mean value at each location as a measurement of the background at that point. When the shadowgraph is
in the linear regime these two methods should be equivalent. The latter method has the advantage that it eliminates slow drifts in the optical system which inevitably accumulate during the period between when the background and convection images are taken. One may estimate the
noise in the digitizing system by computing the \textit{rms} deviation of the divided shadowgraph signal, \( I/I_0 - 1 \). This is shown for a typical driven data set in Figure 3-14 which shows the measured amplitude from the

![Graph](image_url)

\textbf{FIGURE 3-14. Shadowgraph Signal as a Function of Position for Driven Convection}

This figure shows filtered shadowgraph signal for a single image of a 1024 sec time series along with the \textit{rms} amplitude at each location. The \textit{rms} amplitude has been scaled by \( \sqrt{2} \) so that it should coincide with the peaks of the single image shadowgraph. The \textit{rms} has the advantage that it averages out the fast variation of the convection leaving only the amplitude.

The first image in a time series along with the \textit{rms} amplitude computed at each location. The \textit{rms} amplitude has been scaled by \( \sqrt{2} \) so that it may be more easily compared with the peaks of the shadowgraph signal.
This data set was taken with an applied perturbation frequency of 54.69 mHz, a temperature difference of 2.250°C and a Reynolds number of 0.852. The measured amplitude near the inlet is dominated by camera noise which can be estimated to contribute 0.013 bits to the measured shadowgraph signal. In contrast, the same rms data along with the bandwidth limited Fourier data is shown in Figure 3-15. The data is shown on a semilog plot. It is clear that the bandwidth limiting method gives approximately two orders of magnitude better signal to noise ratio than the rms deviation of the signal (4 orders of magnitude in the power).

A similar bandwidth limiting technique can also be applied to the data from undriven convection. There are, however, a couple of complications. First, one may not simply extract the amplitude from a single Fourier bin since there is a range of unstable frequencies. Instead, one must sum the power in a range of bins around the maximally unstable frequency which is found by using the driven measurements. Second, since one can not choose the sampling interval to be commensurate with the drive frequency one should apply a windowing function to the time series before performing the FFT. I use a Hanning window which is a single cycle of a cosine, \( W(t) = [1 + \cos(2\pi t/N\Delta t)]/2 \), where \( N\Delta t \) is the total length of the time series. A time series near the outlet is shown in Figure 3-16. The conditions are the same as the data shown in Figure 3-13 but without the drive applied. The amplitude is much smaller and is no longer constant.
FIGURE 3-15. Relative Amplitude of Driven and Free Convection

This figure shows the measured amplitude of the convection which results when the perturbation heater is on (driven) and when it is turned off (free). Two methods of extracting the amplitude are shown for the driven convection. One method consists of computing the \textit{rms} fluctuation of the shadowgraph signal, the other is the bandwidth limiting method described in the text. It is clear that the Fourier method is nearly two orders of magnitude more sensitive. The perturbation heater is located at \( x = 4 \) where a large bump is visible in the Fourier amplitude for the driven case. It is also clear that the driven convection is at least an order of magnitude larger than the free convection. The drive frequency for this figure was chosen to be near the resonant frequency.

The power spectrum is no longer instrumentally narrow. The power is distributed over a range of bins. The amplitude, computed by taking the square root of the sum of the power in the peak, is shown in Figure 3-15. This method of measuring the power in the signal provides approximately
FIGURE 3.16. Shadowgraph Signal as a Function of Time Near Outlet for Free Convection

This shows the time series of the shadowgraph signal at a location near the outlet of the channel. The amplitude is much smaller than the driven case shown in Figure 3.13. Also the power spectrum is spread out over a range of frequencies. The amplitude shown in Figure 3.15 is computed by taking the square root of the total power in the peak.

an order of magnitude better signal to noise ratio than the $rms$ fluctuation.

3.5 Experimental Determination of Spatial Growth Rate

In this section I will address how to extract the spatial growth rate from the shadowgraph power data. The experimental data collection procedure
for finding the spatial growth rate is to first set a flow and a temperature difference. One then applies the sinusoidal perturbation to the inlet heater and waits for the system to equilibrate. This time scale is set by the time it takes fluid to travel from the inlet to the outlet. Then the amplitude as a function of position is measured using the algorithm described above. The data is fit by a simple growing exponential from the inlet to a position where the amplitude is approximately 10% of the saturation amplitude. By truncating the fit to 10% of the saturation value, one presumably limits the nonlinear saturation terms. The perturbation frequency is changed and the measurement is repeated. The drive frequency is swept through the resonant frequency. The resulting curves for the amplitude as a function of position for a typical data set are shown in Figure 3-17. For the data shown in Figure 3-17, the system has not saturated even at the outlet. Once the spatial growth rate has been extracted, it is plotted against the applied frequency for a series of different temperature differences. The resulting data are shown in Figure 3-18. Since the growth rate can only depend on the magnitude of the frequency, not on its sign, the data for each temperature difference are fit by a quadratic polynomial in \( \omega^2 \). The measurement method can only measure growth rates which are positive or at best slightly negative, the fit function used is

\[
\sigma(\omega) = \frac{1}{2} (a + b \omega^2 + c\omega^4 + |a + b \omega^2 + c\omega^4|) + d. \tag{3-22}
\]
FIGURE 3-17. Shadowgraph Power vs. Position for Driven Convection

This figure shows the shadowgraph power as a function of position for a series of drive frequencies. The data is plotted on a semi-log scale to show the exponential fit which is used to extract the spatial growth rate. Only a subset of the frequencies is shown. This plot shows the shadowgraph power which has twice the spatial growth rate of the amplitude.

The parameter \( d \) measures the noise in the baseline. Where the polynomial would drop below \( d \), the fit function becomes constant. This serves as an interpolating polynomial which can be used to compute the maximal spatial growth rate for each drive frequency which is given by the condition

\[
\sigma_{\text{max}} = a - \frac{b^2}{4c} + d.
\]  

(3-23)
FIGURE 3-18. Spatial Growth Rate vs. Frequency

This shows the spatial growth rate plotted against the frequency of the perturbation heater for several temperature differences.

The maximal growth rate is plotted against the applied temperature difference for the four data sets in Figure 3-19. When the spatial growth rate is small, it is proportional to \( \varepsilon \). Therefore, one may extrapolate to where the growth rate vanishes. This determines the critical temperature difference. In Figure 3-19 the fit is only performed for growth rates less than about 0.2. Above this value, the amplitude rises from the detectability limit to saturation within a few sample points and the fitting shown in Figure 3-17 gives a poor estimate of the growth rate.
FIGURE 3-19. Maximum Spatial Growth Rate vs. Temperature Difference.
This plot shows the height of the peaks in Figure 3-18 as a function of the applied temperature difference. The analysis is replicated for the other three flow rates.

3.6 The Spatial Growth Rate and the Effective Reynolds Number

In this section I will address the question of how to best compare the experimental results for a finite width channel to the infinite width theory. The biggest problem is that the definition of the Reynolds number for the two problems is slightly different. In both cases the Reynolds number is defined as

\[ \text{Re} = \bar{v} d / \nu. \]  

(3-24)

The subtlety comes from the different dependence of the mean velocity on the maximum velocity in the channel for the two cases. In the case of
the infinite aspect ratio \( \nu_{\text{max}} = (3/2)\tilde{\nu} \). So an entirely equivalent definition of the Reynolds number for this geometry could be

\[ \text{Re} = 2\nu_{\text{max}}d/3\nu. \]

However when one includes the lateral boundary conditions, the relationship between \( \nu_{\text{max}} \) and \( \tilde{\nu} \) is substantially more complicated (see Eq A-15 on page 254) so that the "second" definition of the Reynolds number is no longer equivalent to the first. The point is that the group velocity of the rolls depends linearly of the Reynolds number, but the rolls themselves do not travel at either \( \nu_{\text{max}} \) or \( \tilde{\nu} \), but at some velocity between them which depends on the exact geometry of the cell.

I propose that to apply the theory to the experimental results one should define an effective Reynolds number, which is proportional to the measured value, as

\[ \text{Re}^{\text{eff}} = G\text{Re}, \quad (3-25) \]

where \( G \) is a constant which depends on the geometry. One may measure the geometrical constant from measurements of the group velocity, which is proportional to the Reynolds number, by a procedure described below. This definition is somewhat \( \textit{ad hoc} \), but is reasonable in that it effectively resolves the ambiguity of how best to define the Reynolds number. It is, however, probably the weakest part of the data analysis. Fortunately one may make a consistency check of this assumption by asking whether a single value of the \textit{geometrical} constant will work for all of the data sets.
To determine the geometrical constant, consider the CGL when the amplitude is small. One may drop the saturation term and model the growth of the convection with the linearized equation

$$\tau_0 (\partial_t - v_g \partial_x)A = \varepsilon (1 + i c_0)A + \xi_0^2 (1 + i c_1)\partial_x^2 A.$$

(3-26)

When there is a perturbation applied we have the inlet boundary condition $A(0, t) = A_0 e^{-i \omega t}$, where $\omega = \omega_0 - \omega_c$ is the difference between the applied frequency and the critical frequency. Look for a solution in the form of an exponentially growing traveling wave

$$A(x, t) = A_0 e^{i (kx - \omega t) + \kappa t},$$

(3-27)

where $k = q - q_c$ is the difference between the wavevector and the critical wavevector and $\kappa$ is the spatial growth rate. Inserting Eq 3-27 into Eq 3-26 yields coupled algebraic equations for $k$ and $\kappa$.

$$0 = -\varepsilon + \kappa \tau_0 v_g + (k^2 + 2c_1 k \kappa - \kappa^2)\xi_0^2$$

(3-28)

$$0 = -c_0 \varepsilon + (k v_g - \omega)\tau_0 + (c_1 k^2 - 2k \kappa - c_1 \kappa^2)\xi_0^2.$$

Although these equations may be solved exactly, the solution is easier to understand if one makes the approximation that $k$ and $\kappa$ are small compared to one. In that case, the solution takes the simple form

$$k = \frac{\omega}{v_g} + \frac{c_0 \varepsilon}{\tau_0 v_g}$$

(3-29)

$$\kappa = \frac{\varepsilon}{\tau_0 v_g}.$$
The ratio $\varepsilon/\kappa = \tau_0 v_g$ can be used as a direct measurement of the group velocity since $\tau_0$ is effectively constant. The data for this ratio are shown in Figure 3-20. One finds that the group velocity is proportional to the Reynolds number as expected. The slope of the linear fit allows one to define the geometrical constant. The fit to the data gives

$$[v_g \tau_0]^{\text{exp}} = (0.44 \pm 0.01) \text{Re}^{\text{exp}}.$$  \hspace{1cm} (3-30)

The corresponding theoretical result is
\[ [v_g r_0]^{\text{theo}} = 0.4158 \text{Re}. \quad (3-31) \]

So the effective Reynolds number is

\[ \text{Re}^{\text{eff}} = 1.070 \text{Re}. \quad (3-32) \]

It is reassuring that the geometrical correction only amounts to a 7% effect.

3.7 Stabilization of the Conduction State by the Applied Flow

Once the critical temperature difference has been extracted, one may now plot the critical temperature difference as a function of flow rate. Since the critical temperature difference with no flow applied is known to drift with time, I will correct for this drift by computing

\[ \varepsilon_{\text{conv}}(\text{Re}) = \frac{\Delta T_c(Re, t)}{\Delta T_c(0, t)} - 1, \quad (3-33) \]

where \( \Delta T_c(0, t) \) is the critical temperature difference at time \( t \) (when the measurement was performed) with no flow applied as inferred from the fit in Eq 3-10. For small values of the Reynolds number, the suppression of onset is predicted to be quadratic. A fit to the data gives

\[ \varepsilon_{\text{conv}}(\text{Re}) = (0.054 \pm 0.002)\text{Re}^2. \]

However, the theoretical prediction by Müller gives a much smaller suppression \( \varepsilon_{\text{th}}(\text{Re}) = 0.0091\text{Re}^2 \). It is likely that this wide discrepancy is due to the lateral side walls. Indeed this was one of the questions studied in the Ph.D. thesis of J.M. Luijx [6].

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FIGURE 3.21. Convective Onset as a Function of Reynolds Number

The linear stability boundary is plotted against the applied Reynolds number. Also shown is a fit to $\varepsilon_{\text{conv}} = a \cdot Re^2$, where $a$ is an adjustable fit parameter.

Among other things, he computed a Galerkin expansion to determine the suppression of the onset as a function of Reynolds number, the channel horizontal aspect ratio, and the Prandtl number. Without reproducing his work, I have interpolated between the tabulated data in his thesis. It is first important to note that his definition of the Reynolds number differs from this work by a factor of two so that $Re = 2Re_{\text{luijx}}$. Figure 3.22 summarizes the data given in Tables IV.2.b and IV.2.c of Luijx’ thesis which contain the computed critical Rayleigh number as function of Reynolds, $Ra_{\text{crit}}(Re)$ for $\beta = 2$ and $\beta = 5$, $\sigma = 10$. I have converted this data to epsilon and used the definition of the Reynolds number in this
work. In order to compare his data to my cell which has $\beta = 2.25$ and

\[ \sigma = 5.9, \]

I have performed a linear interpolation of the Luijx predictions. The results of this interpolation for $\beta = 2.25$ and $\sigma = 10$ are also shown in Figure 3-22 along with the prediction of Müller et al. for $\beta = \infty$ and $\sigma = 5.8$. The last correction which must be made is to adjust the prediction for the experimental value of the Prandtl number. There are two convection states: transverse rolls, which I have been studying, and longitudinal rolls which occur at a higher applied temperature differences.
The critical temperature difference for longitudinal rolls is independent of the applied through flow. Luijx predicts that the Reynolds number where the critical temperature differences for longitudinal and transverse rolls intersect, scales as $1/\sigma$ when $\sigma \to \infty$. Therefore I will correct for the Prandtl number dependence by scaling the Reynolds number of the prediction above by the ratio of the Prandtl number in the experiment to that of the calculation $\text{Re}' = \text{Re} \sigma_{\text{exp}} / \sigma_{\text{th}}$. The resulting prediction and
data are shown in Figure 3-23. The Luijx prediction agrees quite well

![Graph showing comparison of predicted suppression of onset by flow to experiment.](image)

**FIGURE 3-23.** Comparison of Predicted Suppression of Onset by Flow to Experiment

This figure shows the interpolated and Prandtl corrected prediction of Luijx for the stabilization of the conduction state by the applied flow. The prediction of Müller et. al. for $\beta = \infty$ is included for reference.

with the data.

### 3.7.1 Neutral Curves

From the data in Figure 3-16 one may also extract a measurement of the neutral curve. The fits of the spatial growth rate vs. drive frequency allow one to compute the frequency at which the growth rate vanishes. This is the definition of the neutral curve. The results are shown in Figure 3-24. The neutral curves have been fit by a quadratic polynomial near the min-
FIGURE 3-24. Neutral Curves

This figure shows the frequencies at which the spatial growth rate vanishes. Also shown is the frequency of the maximal spatial growth rate.

imum. In the center of each band is plotted the frequency corresponding to the maximal growth rate as well as the frequency and temperature difference found in Figure 3-19. The quadratic fits have been constrained to go through the critical temperature difference

\[ \Delta T(\text{Re}, f) = A[f - f_c(\text{Re})]^2 + \Delta T_c(\text{Re}). \]  

(3-34)

The only free parameter in the fit is A, which is related to the correlation length, \( \xi_0 \) . These curves can be made nondimensional by converting the temperature difference to epsilon and rescaling the frequency by the critical frequency for that Reynolds number. Define the rescaled frequency
\( \tilde{\omega} = \omega/\omega_c - 1 \). The resulting plot is shown in Figure 3-25. The theory predicts that the frequency is nearly dispersionless, \textit{i.e.} 
\[
\omega(Re) = q(Re) v_\lambda (Re) \left[ 0.991 - 0.00044Re^2 + O(Re^4) \right],
\]
so that to a very good approximation \( \tilde{\omega} \approx q/q_c - 1 \) and the abscissa in Figure 3-25 can be identified with the reduced wavevector which is the more customary way of expressing the neutral curve. By construction, the experimental
neutral curve is constrained to go through \((\varepsilon, \bar{\omega}) = (0, 0)\), so the data are fit by the quadratic function \(\varepsilon = a\bar{\omega}^2\) where \(a\) is the sole fit parameter. From Eq 3-28 one may compute the theoretical value by setting \(\kappa = 0\). One finds

\[
\varepsilon = k^2 \xi_0^2 = k_c^2 \xi_0^2 \bar{\omega}^2.
\]  
(3-35)

Therefore one may extract an experimental measurement of \(\xi_0\) by computing the quantity

\[
\xi_0^{\text{exp}} = \sqrt{a}/k_{\text{exp}} = 0.348,
\]  
(3-36)

where I have used the experimental measurement of the critical wavevector. This can be compared to the theoretical value of

\[
\xi_0 = 0.3850 + 0.0015 \text{Re}^2 + O(\text{Re}^4).
\]  
(3-37)

Again, this discrepancy is probably due to the spiral and the resulting variation of the wavevector across the channel.

3.8 Free Convection

3.8.1 Shadowgraph Calibration

The shadowgraph apparatus is configured to be in the linear regime so that the measured shadowgraph signal is proportional to the amplitude of the temperature field.
\[ \frac{I}{I_0} - 1 = CA, \quad (3-38) \]

where \( C \) is an unknown calibration coefficient. The goal of this section is to determine \( C \). Although it is possible to compute it from the geometry and the known thermal properties of the fluid, it is more precise (and less error prone) to calibrate the shadowgraph apparatus against the theoretical model so that one may convert between the shadowgraph signal and the amplitude. The easiest way is to note that saturated amplitude of steady state convection is

\[ A_{\text{sat}} = \sqrt{\frac{\varepsilon}{g}}. \quad (3-39) \]

If one measures the shadowgraph signal near the outlet, when it is saturated, and plots the square of the shadowgraph vs. \( \varepsilon \), the slope determines the calibration coefficient. This is illustrated by

\[ \left( \frac{I}{I_0} - 1 \right)^2 = C^2 A^2 = C^2 \frac{\varepsilon}{g}. \quad (3-40) \]

Moreover, a time averaged value of \( (I/I_0 - 1)^2 \) is computed when one extracts the spectral power in the peak at the critical frequency. Since the amplitude grows from zero at the inlet to saturation downstream, it is important to only use data where the amplitude is known to be saturated. If the system is too close to the onset, the system will not have saturated by the time the fluid exits the cell. The calibration is easier to perform
when the Reynolds number is small since the spatial growth rate is correspondingly large and the system is saturated near the outlet at a small epsilon. An example of the uncalibrated spectral power \( v \) vs. epsilon for a position near the outlet \((x=56.9)\) is shown in Figure 3-26. The data is fit

\[
Re = 0.300 \\
T_c = 2.554 \\
P = 5.4145 \epsilon
\]

![Graph showing spectral power vs. epsilon](image)

**FIGURE 3-26. Spectral Power in Shadowgraph vs. Epsilon**

This figure shows the time averaged spectral power at position \( x=56.9 \). This position is near enough to the outlet that the convection is near is maximal value without being noticeably perturbed by the outlet. Examination of the amplitude vs. position plot which underlies these data shows that the system for \( \epsilon = 0.0082 \) has not grown to saturation amplitude at the measurement position. On the high end, the amplitude is so large that the video camera is beginning to saturate. Since these two effects skew the measurement, the fit was performed only on the intermediate data.

to the form \( P = a \epsilon \), where \( a \) is an adjustable fit parameter. From Eq 3-40
the calibration constant is seen to be \( C = \sqrt{a g} \). Although the optical system was not readjusted between runs, the shadowgraph calibration was performed separately for each change in the flow rate. This was an attempt to correct for drifts in the optical system.

3.9 Numerical Simulation and Evaluation of the Noise Strength

Given the calibration of the optical system, one may now use the optical system as a quantitative measurement of the local amplitude. This measurement will now be compared to a numerical simulation of the CGL equation. The simulation uses the Crank-Nicholson (CN) forward differences method [7, 8, 9], which I will briefly outline. The CGL equation to be simulated is

\[
\tau_0 (\partial_t + v_g \partial_x) A = \varepsilon (1 + i c_0) A + \xi_0^2 (1 + i c_1) \partial_x^2 A - g (1 + i c_2) |A|^2 A + f(x, t). \tag{3-41}
\]

The CN method consists of approximating the partial differential operators with finite differences which are \textit{implicit} in time and \textit{symmetric} in space. One discretises the field so that \( A_i^n = A(i \Delta x, n \Delta t) \) represents the field \( A \) at spatial location \( i \), at time step \( n \). Then one replaces the linear terms by
\[ A \rightarrow \frac{A_i^n + A_i^{n+1}}{2} \]
\[ \partial_t A \rightarrow \frac{A_i^{n+1} - A_i^n}{\Delta t} \]
\[ \partial_x A \rightarrow \left( \frac{A_{n+1} - A_{n} - A_{n+1} - A_{n-1}}{2\Delta x} \right) \bigg/ 2 \]
\[ \partial_x^2 A \rightarrow \left( \frac{A_{n+1} - 2A_{n} + A_{n-1} + A_{n+1} - 2A_{n+1} + A_{n-1}}{\Delta x^2} \right) \bigg/ 2. \]

The only tricky term is the nonlinear one. Rewrite the nonlinear term as
\[ |A|^2 A \rightarrow \left| \frac{A_i^n + A_i^{n+1}}{2} \right|^2 (A_i^n + A_i^{n+1}) \cdot \] \[ \quad \text{(3-43)} \]

Now, this term is linearized in \( A_i^{n+1} \) so that we can write Eq 3-41 as a linear system. The relevant small parameter is \( \delta \equiv (A_i^{n+1} - A_i^n)/A_i^n \) which should be small everywhere except near defects, where \( A_i^n \) vanishes.

Therefore one may write
\[ \frac{A_i^{n+1} + A_i^n}{2} = \left( 1 + \frac{\delta}{2} \right) A_i^n. \] \[ \quad \text{(3-44)} \]

The nonlinear term can then be written as
\[ |A|^2 A \rightarrow |A_i^n|^2 A_i^n \left[ 1 + \delta + \frac{\delta^*}{2} + O(\delta^2) \right] \]
\[ \rightarrow -\frac{1}{2} |A_i^n|^2 A_i^n + |A_i^n|^2 A_i^{n+1} + \frac{1}{2} (A_i^n)^2 A_i^{n+1*} + O(\delta^2). \] \[ \quad \text{(3-45)} \]

These approximations are first order accurate in time but second order accurate in space. However, the important point is that they are all linear
in $A_i^{n+1}$ and $A_i^{n+1*}$. Rewrite the complex amplitude as two real fields $A_i^n = a_i^n + i b_i^n$ and define staggered real field $\xi_{2i}^n = a_i^n$ and $\xi_{2i+1}^n = b_i^n$. Then one may write the CGL as a linear system

$$\sum_i M_{ij} \xi_i^{n+1} = \nu(\xi_j^n),$$

(3-46)

where $\nu$ is a nonlinear function. This equation can be solved to give the amplitude at time $n+1$, given the amplitude at time $n$. Moreover, the matrix $M_{ij}$ is banded, containing nonzero terms only on the diagonal and the 6 off-diagonal bands. I have employed a variation of $LU$ decomposition, to efficiently solve this matrix equation[7]. Since the matrix is banded, the modified $LU$ decomposition has an algorithmic complexity which scales as $O(N)$ instead of the usual $O(N^3)$ for the general solution to a linear system.

The constants $g$, $\tau_0$, $\xi_0^2$, $\nu_g$, $c_0$, $c_1$, and $c_2$ are actually functions of space since they depend on the Reynolds number which varies along the length of the channel. Also, $\epsilon$ depends directly on the critical temperature difference which in turn depends on the aspect ratio, $\beta$, as was discussed in Section 2.3 on page 47. Therefore the $\beta = \infty$ theory is used to compute all parameters except $\epsilon$, but they are evaluated using the local value of the Reynolds number which, from the geometry of the channel, is inversely proportional to $\beta(x)$. To compute $\epsilon(x)$, one must first compute the local critical temperature difference. The work of Frick and
Clever from which Eq 2-3 is derived, does not include the dependence of the critical temperature difference on the Reynolds number. For lack of a theory which combines both effects, I employed an ad hoc, but reasonable, assumption that the critical Rayleigh number \( Ra_c(\beta, Re) \) can be computed as a combination of the Müller, and Frick and Clever results

\[
Ra_c(\beta, Re) = \frac{Ra_c^{FC}(\beta, 0) Ra_c^{M}(\infty, Re)}{Ra_c(\infty, 0)}, \tag{3-47}
\]

where the superscripts refer to the results of the two theories and \( Ra_c(\infty, 0) = 1707.8 \) is the point were the two theories agree. This is almost certainly not quantitatively correct, but it does have the correct limiting behavior. Fortunately, this expression is only really used to model the variation of the Rayleigh number in the inlet and nozzles. Since the nozzles are only a small fraction of the cell and most of the data is taken near onset where the spatial growth rate is small, the model used for the nozzles doesn’t greatly affect the simulation.

The next issue which must be addressed is the boundary conditions (BCs). This simulation is unusual in that the boundary conditions are almost irrelevant. Since the inlet and outlet regions of the cell are at large negative epsilons, the amplitude there is very small. The BC’s, whether von Neumann, Dirichlet, or constant flux BCs, made no discernable difference to the active region of the cell, so long as the inlet and outlet
regions were made sufficiently long. The saturation amplitude in the inlet, where epsilon is always negative, may be computed analytically. When the Reynolds number is zero it is [10]

\[ A_{\text{sat}} = \sqrt{\frac{f}{2\sqrt{|\varepsilon|}}} \]  

(3.48)

(See APPENDIX D Calculation of Saturated Amplitude for Linearized GL Equation on page 264.) Therefore I fixed the amplitude at this value with a random phase at the inlet. This caused fewer numerical instabilities and allowed the use of a shorter inlet, speeding the calculation. For the outlet I used a zero BC since the physical system didn’t have a long outlet in which the amplitude could adjust smoothly to a small value. In the cell, there was simply a hole in the bottom plate to allow the fluid to exit. This choice of BC seemed more physically relevant. I used a spatial step of \( \Delta x = 0.25 \) and a time step of \( \Delta t = 0.025 \). This leads to Courant conditions of 0.38 and 0.12 on the \( \partial_t A \) and the \( \partial_x^2 A \) terms respectively [9]. The noise strength used in the simulation was computed from the expression given by van Beijeren and Cohen (Eq 1-45 on page 29), who derived the expression for the noise strength of thermal noise driving rolls in an infinite two dimensional plane for rigid-rigid boundary conditions. Since the simulation is stochastic, each run of the simulation yields slightly different results. This makes it very difficult to apply any sort of automatic least squares fitting to the data. Instead I have performed the simulation
using the theoretical value computed from Eq 1-45 which when evaluated using a midplane temperature of $T_{\text{mid}} = 26.25^\circ C$ gives a noise strength of $F_{\text{th}} = 4.4 \times 10^{-9}$. The simulations were run with this value of the noise. The results of the simulation for the four flow rates, along with the experimental data, are shown in Figure 3-27 and Figure 3-28. These figures constitute the primary result.

Consider the data for $Re = 0.300$. The simulation results agree quite well with the data for both small and large temperature differences. The lowest data set shown was taken with $\Delta T_{\text{min}} = 2.550^\circ C$. The data shows no variation of the amplitude with position. This was also true for all smaller temperature differences. This is interpreted as the observational noise floor. Since the simulation amplitude is below the noise floor, this presents no discrepancy. The remaining data sets were taken with temperature increments of $\Delta \Delta T = 0.025^\circ C$ between each set. The results are best interpreted by starting at $\Delta T_{\text{min}}$ and comparing the data to the simulations for successively higher temperature differences. The only notable exception to the otherwise excellent agreement is for $\Delta T = 2.600^\circ C$ where the experimental data does not grow exponentially with distance. The simplest explanation for this anomaly is that the experiment was not run long enough to obtain an accurate ensemble average. In retrospect this is a reasonable interpretation since the simulations show substantial fluctuation from run to run unless the nondimensional time of the simu-
FIGURE 3-27. Comparison of Simulation to Data (I)

This figure shows the experimental amplitude as a function of position, along with a numerical simulation of the CGL. The lowest temperature difference shown is $\Delta T_{\text{min}}$ and the temperature difference between each data set is denoted $\Delta \Delta T$. The results are best interpreted by comparing the lowest data set to its corresponding simulation and working upwards.
FIGURE 3-28. Comparison of Simulation to Data (II)

See Figure 3-27.

lation is much longer than the experiment. Of course it is much easier to let the simulation run for very long nondimensional times that it is to run the experiment for similar times. In order to get good averaging the simu-
lation was run for a nondimensional time of 5000, whereas the experiment is only run for nondimensional times of around 20. If the discrepancy is indeed due to incomplete averaging, one would expect to find the simulation to be sometimes below the data and sometimes above it.

Consider next the data set for Re = 0.489. Again the agreement for the lower temperature differences is quite good. The agreement remains good except for the highest temperature difference $\Delta T = 2.825$. For this case, the simulation grows to near saturation inside the nozzle region. Since the model for the dynamics in the nozzle is not realistic, it is not surprising that the simulation would break down there.

The data set for Re = 0.652 shows somewhat poorer agreement. The simulation and the theory seem to agree well for the large and small temperature differences, but for the intermediate values, the theory is somewhat higher than the data. However, one can note that the theoretical values are fairly uniformly spaced, whereas the experimental data is less uniform. This may again be due to not taking a long enough average to insure an ensemble average. The data set for Re = 0.850 is similar. It agrees well for the small and large temperature difference, but poorly in the middle.
3.9.1 Estimating the Accuracy of the Result

Although the noise strength of $F = 4.4 \times 10^{-9}$ is consistent with the data, no estimate has yet been made of how accurately this value can be determined. In order to estimate the accuracy of this result two additional simulation results will be presented. In the first the noise in the simulation was enhanced by an order of magnitude, in the second the simulation noise was reduced by an order of magnitude. The results for Re = 0.489 are shown in Figure 3-29. The simulations for the other data sets give similar results. It is clear that the simulation no longer fits when the noise is changed by an order of magnitude. The simulation with 10 times thermal noise is consistently larger than the measured data. Similarly the simulation with 0.1 times thermal noise, is consistently lower than the measured data except for the highest applied temperature difference where the simulation grows to near saturation in the inlet nozzle where, as discussed earlier, the simulation is not reliable. The most noticeable effect of changing the noise strength in the simulation is to change the saturation value of the amplitude in the inlet region. One may think of this value as a boundary condition for the simulation in the active region of the cell. Indeed, the fact that the amplitude curves for the different temperature differences all extrapolate to a constant value around $A = 10^{-4}$ which is consistent with Eq 3-47 suggests that a single value of the noise can account for all of the experimental data sets.
FIGURE 3-29. Comparison of Simulation to Data for Large and Small Noise

This figure shows the results of two simulations of the CGL. The upper plot has an applied noise of $F = 10F^{th}$ and the lower plot has an applied noise $F = 0.1F^{th}$. In the upper figure it is clear the simulation is consistently larger than the measured data. In the lower figure the simulation is systematically too small, except for the largest applied temperature difference where the system begins to saturate in the inlet and the model breaks down.
The simulation results for these bracketing cases clearly yields a poorer fit to the data than thermal noise. However, these are clearly well beyond what are required to degrade the fit. One may say that the results of the simulation agree with the empirical data to within a factor of 5, possibly as good as a factor of 2 or 3. There are enough systematic errors that this agreement is quite remarkable.

3.10 Absolute Stability Boundary

It was discovered by Babcock et. al. that there is a simple phenomenological criterion which can be used to find the absolute stability boundary [11]. In the convectively unstable state, the system is noise driven. When the system crosses the absolute stability boundary, it becomes nearly insensitive to noise. They found that if one takes a time series of the amplitude far enough downstream, where it is saturated, the system shows phase noise when convectively unstable, but none when absolutely unstable. This manifests itself as a decrease in the second moment of the power spectrum.

The second moment is artificially broadened by the windowing function applied to the time series before the power spectrum is computed. This causes a purely sinusoidal time series to have a second moment of

\[ \sigma = \sqrt{1/3} \]. (See APPENDIX F The Effect of Windowing on DFT Power

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Since I wish to investigate the deviation of the amplitude from perfect periodicity, I will compute an effective moment given by \( \sigma_{\text{eff}} = \sigma - \sqrt{1/3} \) so that the effective moment vanishes when the system is perfectly periodic. Figure 3-30 shows the effective second moment as a function of position for various temperature differences. The second moment is measured in Fourier bins. Since the sample covered a period of 1024 sec, each Fourier bin represents a frequency interval of 1/2048 Hz. In terms of the vertical diffusion time, this is a dimensionless...
less frequency resolution 0.069. The second moment fluctuates wildly near the inlet where the amplitude is small and the measurement is dominated by the noise. One should only consider the data downstream where the amplitude has saturated. For all the data shown in Figure 3-30, the amplitude saturates beyond about $x = 10$. The dashed lines show the data for the temperature difference below the absolute stability boundary. Below the boundary the second moment is relatively large and varies with position. As one approaches the boundary, the numerical value of the moment decreases until above the boundary the peak collapses to become instrumentally narrow, and no longer varies with position downstream. This is best seen in Figure 3-31 which shows the second moment near the outlet boundary as a function of the applied temperature difference.

Above the absolute boundary the effective second moment drops to less than $10^{-3}$. The peak remains instrumentally narrow for all temperature differences above the transition. The criterion used to assign the absolute stability boundary is when $\sigma_{\text{eff}}$ falls below an arbitrarily chosen threshold of $10^{-2}$. The boundary is then assigned to a value halfway between the two measurements which bound it. The $\varepsilon$ step size is used as an estimate of the uncertainty of the measurement. One must be a little careful about assigning the transition temperature based on the first temperature difference which shows a small second moment. It sometimes happens that a fluctuation of the convectively unstable data will cause the peak to
FIGURE 3-31. Effective Second Moment Near Outlet vs. Epsilon

This figure shows the effective second moment at the dimensionless position \( x = 59 \) as a function of the temperature difference. This position is chosen to be as close to the outlet as possible but not so close that it is perturbed by the outlet hole.

become narrow for that one run at that one position. In these cases examination of the full plot of width vs. position shows that the moment is not independent of position. Figure 3-32 show a summary of the results for the four flow rates. The run for \( \text{Re} = 0.850 \) didn't go to a large enough \( \varepsilon \) to find the boundary. Therefore the last point in that run is used as a lower bound as illustrated by the large error bar. The upper curve is a theoretical prediction for the absolute boundary given by the condition

\[
\varepsilon_{\text{abs}} = \varepsilon_{\text{conv}} + \frac{(v_{\alpha}^2 \tau_0)^2}{4(1 + c_1^2) \xi_0 \tau},
\tag{3-49}
\]
FIGURE 3-32. Absolute and Convective Stability Boundaries.

This figure summarizes the measurements of the stability boundaries as a function of Reynolds numbers along with the theoretical predictions. The data for the convective boundary along with the theoretical prediction by Luijx are shown for reference (See Figure 3-23). The upper line is the Müller prediction for the absolute stability boundary using the onset given by Luijx.

where each term is a function of the Reynolds number. When evaluating Eq 3-49, I used the Luijx value for $\varepsilon_{\text{conv}}$ since it is the more physically realistic value. I could have used the experimentally determined values, but since the prediction and experiment agree, the results would have been qualitatively similar, except that the resulting hybrid theory would not be smooth and it is clear that the correct theory would be. Substituting in the known Reynolds dependence of the parameters produces the theoretical prediction

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\[ \varepsilon_{abs} = \varepsilon_{\text{conv}}^{\text{Luijx}}(\text{Re}) + 0.2918 \text{Re}^2 - 0.0016 \text{Re}^4 + O(\text{Re}^6). \quad (3-50) \]

The agreement between the measurement and the prediction is remarkable. It should be stressed that this is not a fit. There are no adjustable parameters.
CHAPTER 3

Bibliography


4.1 Introduction

The shadowgraph method is one of the most popular methods of visualizing convection patterns. It is simple to implement and provides a noninvasive measurement of the lateral index of refraction modulation within a cell. The situation first considered is illustrated in Figure 4-1, "Experimental Setup." A parallel beam of light is incident from below at an angle

FIGURE 4-1. Experimental Setup
\( \theta \) to the normal. The convection rolls lie in the \( x-y \) plane and are aligned with the \( y \) axis. I will use \( x_0 \) as the \( x \) coordinate where a specific ray enters the cell on the bottom plane and \( x_1 \) as the \( x \) coordinate where the ray exits the top of the cell. The wavelength of the pattern is \( \Lambda \), and the thickness of the layer is \( d \). The wavelength of the light in vacuum is \( \lambda \). Assume that the index of refraction for convecting fluid is given by the general functional form

\[
n(x, z) = n_0 + n_1 a(x) b(z) + n_2 \frac{z}{d}.
\]  

(4-1)

where \( n_1 \ll 1 \) and \( n_2 \ll 1 \) represent the index modulation produced by the convective temperature field and the conduction temperature gradient respectively. The function \( b(z) \) is the vertical eigenmode of the convective temperature field and vanishes on the two surfaces, \( z = 0 \) and \( z = d \). For weak convection it will be taken as \( b(z) = \sin(K'z) \), where \( K' = \pi/d \). The function \( a(x) \) is the horizontal index modulation which is periodic, with period \( \Lambda \). For straight rolls of weak convection one may take \( a(x) = \cos(Kx) \), where \( K = 2\pi/\Lambda \).

The remainder of this chapter will consist of several sections. In the first section I will review the geometrical optics theory of the shadowgraph and discuss its limitations. In the second section I will calculate the shadowgraph signal which results from a pattern of straight rolls. The technique used is based on a method known as the eikonal approximation [1].
This will be done both for light polarized along the roll axis and perpendicu-
lar to the roll axis. It will be shown that for experimentally relevant
parameters, these two situations produce the same results. Then the effect
of the constant gradient of the index in the z-direction will be considered.
It will be shown that even relatively large gradients \( n_1 \ll n_2 \ll 1 \) have only
a very small effect on the resulting image. Then I discuss the effect of illu-
minating the cell with a beam incident at a small angle \( \Theta \) to the normal.
This will be used to estimate the smearing of the image which results
from the use of non-ideal light sources. This is necessary since real instru-
ments only generate an approximation to a plane parallel beam. If the
light source used to illuminate the sample is too large, this smearing can
be substantial. Next I will discuss the application of this theory to super-
positions of many rolls. I will show that in the weakly diffracting limit,
this results in the product of the shadowgraph signals of the individual
component rolls. In the last section I will compute the shadowgraph pat-
tern which results from electroconvection in a nematic liquid crystal. The
liquid crystal has a different focusing mechanism. The material is anis-
tropic and acts as a uniaxial crystal. Unlike thermal convection which
modulates the magnitude of the index of refraction, electroconvection
convection preserves the magnitude of the index of refraction but modu-
lates the direction of the optical axis. This will be compared to the ther-
mal convection case.
It should be noted that the shadowgraph problem is closely related to the study of light scattering from ultrasonic waves [1]. The shadowgraph problem differs primarily in the physical origin and the spatial pattern of the index of refraction modulation which scatters the light. With the shadowgraph method, one is usually interested in the real space intensity pattern, whereas for the ultrasound problem, most measurements are performed in the Fraunhofer limit where one measures the intensity of each diffracted beam separately.

4.2 Geometrical Optics

Lateral variations in the refractive index of the fluid bend the light rays. This results in intensity variations across the beam after it exits the fluid. These intensity variations are straightforward to interpret within the context of geometrical optics [2]. Light rays are bent toward regions of higher refractive index. The angular deflection of a ray is proportional to the lateral gradient of the index of refraction. The rolls therefore act as weak lenses and the light focuses above the cold regions. The warm regions are correspondingly dark. More precisely, one can use Fermat's principle of least time to compute the ray paths.
4.2.1 Computation of the Ray Path

Fermat’s principle of least time states that a light ray will take the path which extremizes the time of propagation. The local wave speed in the material is \( v = c/n \). Therefore the time required for a ray to propagate a distance \( ds \) along a path is given by \( dt = ds/v \). To compute the path one must solve

\[
0 = \delta \int dt = \delta \int \frac{n}{c} \, ds. \tag{4-2}
\]

Write the path \( x(z) \) as a function of \( z \), so that \( ds \) can be written as

\[
ds = \sqrt{1 + x'(z)^2} \, dz.
\]

This yields the Euler-Lagrange equation

\[
\frac{\partial f}{\partial x} - \frac{d}{dz} \left( \frac{\partial f}{\partial x'} \right) = 0. \tag{4-3}
\]

where \( f(x, x', z) = n(x, z) \sqrt{1 + x'^2} \). I will seek a solution as a power series in the small parameter \( n_1 \). Expand the ray path as

\[
x(z) = x_0(z) + x_1(z) n_1 + x_2(z) n_1^2 + O(n_1^3). \tag{4-4}
\]

The boundary conditions on the bottom surface of the cell are

\[
x_0(0) = x_0, \quad x_0'(0) = \tan^{-1} \theta, \quad \text{and} \quad x_i(0) = x_i'(0) = 0 \quad \text{for} \quad i \neq 0.
\]

Insert the series expansion Eq 4-4 into Eq 4-3 and set the coefficients of each power of \( n_1 \) separately to zero. One finds the \( O(1) \) ODE is

\[
\left( n_0 + n_2 \frac{z}{d} \right) x_0''(z) + n_2 \frac{d}{d} x_0'(z) [1 + x_0'(z)^2] = 0. \tag{4-5}
\]

The \( O(n_1) \) ODE is
\[ (n_0 + n_2 \frac{\theta}{2})x_1''(z) + a(x_0(z))b(z)x_0''(z) \\
+ \frac{b(z)}{2}x_1'(z)(1 + 3x_0'(z)^2) - b(z)a'(x_0(z))x_0'(z)^2 \\
+ a(x_0(z))b'(z)x_0'(z)(1 + x_0'(z)^2) \\
- b(z)a'(x_0(z)) = 0. \]  

(4-6)

The \( O(n_1^2) \) ODE is

\[ (n_0 + n_2 \frac{\theta}{3})x_2''(z) + a(x_0(z))b(z)x_1''(z) \\
+ b(z)x_1'(z)a'(x_0(z))x_0''(z) \\
- b(z)x_1(z)x_0'(z)^2a''(x_0(z)) \\
- b(z)x_1(z)a''(x_0(z)) + 3 \frac{b(z)}{2}x_0'(z)^2x_2'(z) \\
+ \frac{b(z)}{3}x_2'(z) + 3 \frac{b(z)}{2}x_0'(z)x_1'(z)^2 \\
+ 3a(x_0(z))b'(z)x_0'(z)^2x_1'(z) \\
- 2b(z)a'(x_0(z))x_0'(z)x_1'(z) \\
+ a(x_0(z))b'(z)x_1'(z) \\
+ x_1(z)a'(x_0(z))b'(z)x_0'(z)(1 + x_0'(z)^2) \\
= 0. \]  

(4-7)

To simplify the analysis, first solve the restricted problem of \( n_2 = 0 \) and normal incidence, \( \theta = 0 \), and then consider the general problem as a modification of the restricted problem. The terms independent of \( n_1 \) yield an ODE for \( x_0(z) \),

\[ x_0''(z) = 0. \]  

(4-8)

This is trivially solved by \( x_0(z) = x_0 \). Insert this into Eq 4-3 and collect the \( O(n_1) \) terms to get an ODE for \( x_1(z) \)

\[ b(z)a'(x_0) - n_0x_1''(z) = 0. \]  

(4-9)

The solution which satisfies the boundary conditions is
\[ x_1(z) = \frac{1}{n_0} a'(x_0) \int_0^z (z - z') b(z') dz'. \quad (4-10) \]

Similarly one finds the solution for \( x_2(z) \) as

\[ x_2(z) = \frac{1}{n_0} \int_0^z (z - z') f(z') dz', \quad (4-11) \]

where

\[
f(z) = -a(x_0) b'(z) x_1''(z) + b(z) x_1(z) a''(x_0) - a(x_0) b(z) x_1''(z).
\quad (4-12)
\]

4.2.2 Focal Length of Convection Rolls

We can estimate the focal length of the convection lenses by considering the simplified problem of Figure 4-2, which depicts a ray passing through

![Figure 4-2. Derivation of Focal Length](image-url)
a convection cell. In this simplified problem take \( b(z) = \sin(K'z) \) and assume that the medium outside the cell has the index of refraction \( n_{\text{air}} \). The ray enters the cell at \( x_0 \) and exits the cell at \( x_1 \). On a plane a distance \( z \) above the cell the beam passes through

\[
x_2 = x_1 + (z - d) \tan \theta_{\text{air}}.
\]

(4-13)

With the equations in the previous section the ray path inside the material can be computed to be

\[
x(z) = x_0 + \frac{n_1}{n_0} \frac{K}{K'} \sin(Kx_0) \[ \sin(K'z) - K'z \] + O(n_1^2).
\]

(4-14)

The exit angle inside the top boundary is given by

\[
\tan \theta_1 = x'(d) = -\frac{2n_1}{\pi n_0} K d \sin(Kx_0) + O(n_1^2).
\]

(4-15)

Correcting for the Snell's law bending at the boundary we find that the angle in the air above the medium is given by

\[
\tan \theta_{\text{air}} = -\frac{2n_1}{\pi n_{\text{air}}} K d \sin(Kx_0) + O(n_1^2).
\]

(4-16)

Therefore one may rewrite Eq 4-13 as

\[
x_2 = x_0 - \frac{2n_1}{n_{\text{air}}} \frac{K}{K'} (z - z_0) \sin(Kx_0) + O(n_1^2),
\]

(4-17)

where \( z_0 = d(1 - n_{\text{air}}/2n_0) \). Note that \( z_0 \) plays the role of the effective optical origin of the shadowgraph effect. If one images the plane \( z = z_0 \) onto a screen, no shadowgraph focusing will be observed. If the medium
outside of the cell has the same mean index as inside the cell so that there is no Snell’s law bending at the top surface, \( i.e. if n_{\text{air}} = n_0 \) then \( z_0 = d/2 \). The shadowgraph signal vanishes at the center of the cell and not on the top surface as was found in previous work by Rasenat et al [3]. However, if the focal length, defined by the distance to the point where nearby rays first cross, is large compared to the cell thickness, \( z_0 \) is only a small correction. The intensity which appears on the image plane is given by conservation of energy (c.f. Eq 4-33) as

\[
I(x_2)dx_2 = \sum_{\text{rays}} I_0 dx_0 ,
\]

(4-18)

where the sum is to be taken over all rays which pass though the point \((x_2, z)\). To determine the number of rays, consider that Eq 4-17 is a single valued function which computes \( x_2 \) as a function of \( x_0 \), \( i.e. \)

\( x_2 = g(x_0) \). The inverse function, \( x_0 = g^{-1}(x_2) \), is in general multivalued. Each inverse corresponds to one term in the sum in Eq 4-18. \( I_0 \) is the uniform intensity of the illumination. The small area elements \( dx_0 \) and \( dx_2 \) are related by \( dx_2 = g'(x_0)dx_0 \). One may rewrite Eq 4-18 as

\[
I(x_2) = I_0 \sum_{\text{rays}} \frac{1}{g'(x_0)} = I_0 \sum_{\text{rays}} \frac{1}{g'(g^{-1}(x_2))}.
\]

(4-19)

If one only considers image planes close enough to the cell that none of the rays cross, \( x_0 = g^{-1}(x_2) \) is single valued. Further from the cell, one encounters caustics, points where the rays cross. In geometrical optics
caustics are unphysical points which have an infinite intensity. They are defined by the zeroes of $g'(x_0)$. As one passes the first caustic, the sum may have one or three terms depending on the value of $x_2$. This is illustrated schematically in Figure 4-3 and Figure 4-4.

![Figure 4-3. Ray Displacement in Geometrical Optics.](image)

This figure shows a plot of Eq 4-17. The ordinate is the $x$ position of ray as it crosses a plane a distance $z$, from the cell. The abscissa is the $x$-coordinate of a ray as it enters the bottom of the cell. The plot shows several different image plane distances. This function must be inverted to compute the number of rays which contribute to the sum in Eq 4-19. When $z$ is less than a critical distance $f$, the function $x_0 = g^{-1}(x_2)$ is single valued and only one term contributes.

Define the effective focal length $f$ as the distance from the cell to the nearest caustic. For $z < f$, there is only one term in the sum Eq 4-19, and
FIGURE 4-4. Geometrical Optics Ray Paths Beyond the Cell

This figure schematically shows the propagation of the light rays after they leave the cell. Before the first caustic, the rays have not crossed and Eq 4-15 has only one term. Between the first and second caustics, the sum will require one or three terms. The lines of caustics are the v shaped crossing lines which radiate from the focal points.

\[
I(x_2, z) = I_0 \left[ 1 - 2 \frac{n_1}{n_{\text{air}}} \frac{K^2}{K} (z - z_0) \cos(Kg^{-1}(x_2)) \right]^{-1}. \quad (4-20)
\]

This expression also allows one to compute the lines of caustics. Whenever the right hand side diverges it dominates the sum Eq 4-19 and a caustic appears. This is shown in Figure 4-5 which shows that the caustics extend to infinity. Clearly, convection lenses have strong aberrations!
FIGURE 4-5. Lines of Caustics.

This figure shows the lines of caustics for a cell with d=5 mm, L=1 cm, \( n_0 = 1 \), \( n_1 = 10^{-3} \). The caustics appear when rays shown in Figure 4-4 cross. They extend from \( z = f \), to infinity.

Up to the first caustic, the intensity is maximal along the line \( x = 0 \) which is above the cold, high density regions. Along this line, Eq 4-20 has a simple analytic form since \( g(0) = 0 \). The intensity is given by

\[
I(0, z) = I_0 \left[ 1 - 2 \frac{n_1}{n_{\text{air}}} \frac{K^2}{K} (z - z_0) \right]^{-1}.
\]  

(4-21)

This implies that the focal length is \( f = n_{\text{air}} \lambda^2 / (8\pi n_1 d) + z_0 \). An important observation is that this expression predicts that the focal length diverges as the index modulation vanishes, \( n_1 \to 0 \). When \( n_1 \) is small one usually wishes to achieve the largest image contrast, and adjusts the shadowgraph apparatus to image a plane near the focus of the rolls. No
finite diameter lens can focus sharply on a plane infinitely far away.

Clearly, at some distance from the cell, the image formed by the convection lenses will be limited by diffraction. A simple qualitative argument allows one to estimate when this occurs.

A diffraction limited lens the same size and focal length as a convection roll creates a spot which is roughly \( \lambda f / (\frac{\Lambda}{2}) \) in diameter. When this becomes comparable to the roll size \( \Lambda/2 \) the system is clearly diffraction limited. So we have a characteristic length \( f_d \) defined by

\[
\frac{\Lambda}{2} \sim \lambda f_d / (\Lambda/2)
\]

at which diffraction dominates the focusing. Solving yields

\[
f_d = \frac{\Lambda^2}{(4\lambda)} .
\]

(4-22)

Since convection roll lenses are hardly diffraction limited, one would expect geometrical optics to be valid only for viewing image planes which are much closer to the cell than \( f_d \).

Another useful limit is obtained when one considers the images which form near the cell where \( z \ll f \). The rays deviate little from where they exit the cell. In this limit \( x_2 = x_1 + O(n_1) \frac{\Lambda^2}{d} \) so that

\[
\rho^{-1}(x_2) = x_2 + O(n_1) \frac{\Lambda^2}{d}
\]

and Eq 4-20 can be written as

\[
I(x_2, z) = I_0 \left[ 1 + \frac{2}{\pi} \frac{n_1}{n_{air}} K^2 d (z - z_0) \cos(K x_2) + O(n_1^2) \right].
\]

(4-23)
This is sometimes called the linear regime since $I(x_2, z)/I_0$ grows linearly with $z$ and $n_1$.

The geometrical optics treatment has two main deficiencies. It predicts that the rays will focus to infinite intensity on the caustics and that the focal length diverges as the index modulation vanishes. Diffraction always limits the focalization, and thus prevents the formation of caustics. In the vicinity of the caustics one must use physical optics for a quantitative prediction of the intensity. More importantly, geometrical optics is completely inapplicable when one images distances from the cell approaching $f_d$ which is the case when the index of refraction modulation is weak or the cell is thin. For these cases the shadowgraph image is limited by diffraction.

4.3 Physical Optics

The physical optics picture of the shadowgraph has a somewhat different interpretation than does geometrical optics. Plane waves travelling through the medium are subject to variations in the local wave speed, $c/n$. Wavefronts in regions with a higher $n$ propagate more slowly than regions with a lower $n$. This causes the phase of the wavefront to be locally retarded or advanced. When the wave emerges from the cell, the amplitude is nearly unchanged, but the phase of the wavefront has been
spatially corrugated, and the cell acts as a diffraction grating as shown in Figure 4-6. Once the wave exits the cell, the electric field can be propagated through the air above the cell by Fresnel diffraction. Squaring the field gives the intensity pattern which will appear on a screen placed at any particular distance from the cell. It is easy to estimate the distance to the first maximum of intensity in the limit that the phase difference across the beam is small. This is the physical optics analogue of the focal length which appears in the geometrical optics theory. If the index modulation is very weak, the diffraction grating scatters light into only the first order diffraction beams which travel at plus and minus the diffracted angle given by \( 2k \sin(\theta_d/2) = K \). Since there is no intensity modulation on the cell surface, the sum of the diffracted beams must emerge from the cell 90° out of phase with the main beam. When the diffracted beams
propagate a distance $\lambda/4$ further than the main beam, they become in phase with the main beam and the intensity modulation is maximal. This is illustrated schematically in Figure 4-7.

![Diagram](image.png)

**FIGURE 4-7. Physical Optics Focal Length**

The first intensity maximum occurs when the diffracted beams interfere constructively with the main beam. This occurs when they travel a distance $\lambda/4$ further than the main beam since they exit the cell $90^\circ$ out of phase.

The focal length is therefore given by the criterion $f(\sec \theta_d - 1) = \lambda/4$ which implies $f = \lambda^2 / 2\lambda$. The focal length for weak gratings is independent of the strength of the index modulation, $n_1$. This is in stark contrast to the prediction of geometrical optics. It is primarily in this limit that this theory will be most useful.
4.3.1 The Wave Equation and the Eikonal Approximation

I will begin by reviewing the eikonal approximation [1] and its application to the shadowgraph method. This method is based on a short wavelength approximation to the full wave equation. Begin with Maxwell's equations in a nonconducting, nonmagnetic medium with no free charges.

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \]
\[ \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{D} = 0. \]  
(4-24)

Since the medium is nonmagnetic, \( \mu = 1 \), and \( \mathbf{H} = \mathbf{B} \). Eliminate \( \mathbf{H} \) from the first line, by taking the curl of the first equation, to get

\[ \nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{D}}{\partial t^2}. \]  
(4-25)

Recall the vector identity, \( \nabla \times \nabla \times = -\nabla^2 + \nabla(\nabla \cdot) \) and the constitutive relation \( \mathbf{D} = \varepsilon \mathbf{E} \) where \( \varepsilon = n^2 \) to get the wave equation

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\varepsilon \mathbf{E}) = \nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}). \]  
(4-26)

The last term can be rewritten by using \( 0 = \nabla \cdot \mathbf{D} = \mathbf{E} \cdot \nabla \varepsilon + \varepsilon \nabla \cdot \mathbf{E} \) which implies \( \nabla \cdot \mathbf{E} = -\mathbf{E} \cdot \nabla \varepsilon / \varepsilon \). The convection rolls lie with their axes along the \( y \) direction. Their index of refraction is independent of \( y \) so \( \nabla \varepsilon \) lies in the \( x-z \) plane. When the incident light is polarized in the \( y \) direction, \( \mathbf{E} \) and \( \nabla \varepsilon \) are orthogonal and this term vanishes. For the \( x \) polarization one can compare the magnitude of the terms in Eq 4-26.
The magnitude of the last term is $|\nabla(\nabla \cdot E)| \sim n_1 k K/n_0$ and $|\nabla^2 E| \sim k^2$ so $\nabla(\nabla \cdot E)$ is $n_1 K/k$ times smaller than the $\nabla^2 E$ term and will be neglected. It should be noted that in Eq 4-26, only the $\nabla(\nabla \cdot E)$ term mixes components of the electric field. The other terms act independently on each scalar component. Dropping it is equivalent to ignoring changes in the direction of the polarization vector. This implies that the shadowgraph images produced by both polarizations are equivalent within this approximation. One can rewrite Eq 4-26 as a scalar wave equation by writing the complex electric field as

$$E(x) = E(x)e^{i(k\phi(x) - \omega t)}$$  \hspace{1cm} (4-27)

where $E(x)$ and $\phi(x)$ are real scalar functions of position. $\omega$ is the frequency of the light and $k = \omega/c$. $\vec{E}$ is the polarization vector which may be complex, but is not a function of position. $E(x)$ represents the local amplitude of the field and should be a slowly varying function of space and $\phi(x)$ is the local phase and is called the eikonal. Insert this ansatz into Eq 4-26 and match the real and imaginary parts of the resulting expression to get

$$\frac{\omega^2}{c^2} e(x) = k^2 |\nabla \phi(x)|^2 - \nabla^2 E(x)/E(x), \hspace{1cm} (4-28)$$

$$0 = 2\nabla E(x) \cdot \nabla \phi(x) + E(x) \nabla^2 \phi(x). \hspace{1cm} (4-29)$$

The last term in Eq 4-28 is smaller than the others by a factor of $(k/K)^2$. If we assume that $k/K$ is large, this term can be neglected. This is the small
wavelength approximation which is the basis of geometrical optics. It relies on the assumption that there is no signification variation of the amplitude $E(x)$ on the scale of a wavelength. Eq 4-28 can be rewritten as

$$n(x) = |\nabla \phi(x)|,$$

(4-30)

where $n(x)$ is the local index of refraction. The geometrical ray is defined as the direction of the Poynting vector, $E \times H^*$, which is aligned normal to the lines of constant phase. Therefore, Eq 4-30 states that the derivative of the phase along the geometrical ray is equal to the local index of refraction. Define a new coordinate $s$, as the distance along the ray, and rewrite Eq 4-30 as

$$n(x) = \frac{d\phi}{ds},$$

(4-31)

which is to be evaluated along the ray path. If the ray path, parametrized by $s$, is denoted $x(s)$, the solution to Eq 4-31 is then

$$\phi(s) = \int_0^s n(x(s'))ds'.$$

(4-32)

This states that the eikonal is simply the phase which accumulates along the path of the geometrical ray. To compute it, one must solve for the ray path and integrate the index of refraction along the ray path to get the eikonal.
I have outlined the procedure for computing the eikonal. To complete the derivation one also needs the amplitude $E(x)$. Multiply Eq 4-29 by $E(x)$ and write it as

$$\nabla \cdot (E^2 \nabla \phi) = 0.$$  \hfill (4-33)

Denote the unit vector along the ray path $\mathbf{x}'(s) \equiv \mathbf{s}$ and write $\nabla \phi = n\mathbf{s}$ so that Eq 4-33 becomes

$$\nabla \cdot (E^2 n\mathbf{s}) = 0.$$  \hfill (4-34)

Consider a gaussian surface around an infinitesimal volume aligned with a local bundle of rays as illustrated in Figure 4-8. The surface is chosen so that the sides of the surface are tangent to the local rays. The end caps are perpendicular to the rays.

![Figure 4-8: Gaussian Surface](image)

The sides of the surface are chosen to lie parallel to a bundle of rays. The end caps are perpendicular to the rays.

that the sides of the surface are tangent to the local rays. The end caps are chosen to be sufficiently small that the density of rays is constant across the surface $A(s)$. The definition of the divergence of a vector field implies

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\[
\lim_{V \to 0} \int_{\text{surface}} E^2 n \hat{s} \cdot dA = 0. \quad (4-35)
\]

The integral vanishes on the sides of the surface since \( \hat{s} \) and \( dA \) are orthogonal. On the end caps, \( s \) and \( dA \) are parallel and we are led to the conclusion

\[ A(s)E^2(s)n(s) - A(s + ds)E^2(s + ds)n(s + ds) = 0. \quad (4-36) \]

This expression can be integrated for an arbitrary distance along \( s \) which implies \( A(s)E^2(s)n(s) \) is constant along the ray. To compute the amplitude \( E(s) \), the problem is reduced to computing the change in an area element perpendicular to the rays. This is straightforward if one has previously computed the ray paths.

The procedure outlined to compute the amplitude and phase modulation of the electric field uses the concept of a ray path. However, it must be stressed, this approach is not equivalent to geometrical optics. It yields the correct diffraction behavior for the limit that the bending of the wavefront occurs on a length scale large compared to the wavelength of light.

4.3.2 Computation of the Eikonal as a Series Expansion in the Index Perturbation

To compute the eikonal, one must first compute the ray paths through the cell. Insert Eq 4-1 and Eq 4-4 into Eq 4-32 to get
\[ \phi(z) = \int_0^z \left( n_0 + n_2 \frac{z'}{d} \right) \left[ 1 + x_0(z')^2 \right]^\frac{1}{2} dz' + n_1 \int_0^z \left\{ a(x_0(z')) b(z') \left[ 1 + x_0(z')^2 \right]^\frac{1}{2} \right. \\
+ \left. \left( n_0 + n_2 \frac{z'}{d} \right) x_0(z') x_1(z') \left[ 1 + x_0(z')^2 \right]^\frac{1}{2} \right\} dz' + n_1^2 \int_0^z \left\{ b(z') x_1(z') a'(x_0(z')) \left[ 1 + x_0(z')^2 \right]^\frac{1}{2} \right. \\
+ \left. a(x_0(z')) b(z') x_0(z') x_1(z') \left[ 1 + x_0(z')^2 \right]^\frac{1}{2} \right. \\
+ \left. \frac{1}{2} \left( n_0 + n_2 \frac{z'}{d} \right) x_1(z')^2 + 2 x_0(z') x_2(z') \right. \\
+ \left. 2 x_0(z')^2 x_2(z') \left[ 1 + x_0(z')^2 \right]^\frac{1}{2} \right\} dz' \\
+ O(n_1^3). \] (4-37)

In the next section (see Eq 4-8) I will show that for the case of normal incidence, \( x_0(z) \) is constant, so that \( x_0'(z) = 0 \) and Eq 4-37 takes a much simpler form

\[ \phi(z) = n_0 z + \frac{n_2 \frac{z^2}{d}}{2} + n_1 a(x_0) \int_0^z b(z') dz' + n_1^2 \int_0^z \left[ b(z') x_1(z') a'(x_0) + \frac{1}{2} \left( n_0 + n_2 \frac{z'}{d} \right) x_1(z')^2 \right] dz' + O(n_1^3). \] (4-38)

The first line of Eq 4-38 represents the mean phase accumulated by the wave as it traverses the cell; it has no variation across the wavefront and does not contribute to the shadowgraph image. The second line is the variation of the phase across the wavefront which would accumulate if the
rays passed straight through the material without bending. The $z$ dependence appears as a simple average of $b(z)$ through the cell. The last line gives the first correction due to the ray bending. To get the eikonal to $O(n_1^2)$, one need only compute $x_1(z)$. This expansion may be truncated when the terms of $k\psi(x)$ become small compared to $2\pi$. Note that each term grows with $z$, so even though the terms in this expansion are decreasing as powers of $n_1$, for very thick cells, higher order terms will become increasingly important. I will, however, only carry this expansion to $O(n_1^2)$.

One can compute the amplitude by considering the ray path evaluated on the two boundaries; $x_0 \equiv x(0)$ and $x_1 \equiv x(d)$. Neighboring rays a distance $dx_0$ apart on the bottom surface emerge a distance $dx_1 = \frac{\partial x_1}{\partial x_0} dx_0$ apart on the top surface. Equation 4-36 implies that the amplitude on the top surface is given by

$$E_1(x_1) = E_0 \sqrt{\frac{n_0}{n_0 + n_2} \frac{dx_0}{dx_1}}. \quad (4-39)$$

This assumes that the angle through which the ray is bent is small so that one may take $x_1$ to be perpendicular to the ray path on the top surface in the computation of the area element across the beam.
4.3.3 Propagation of the Electric Field Beyond the Cell.

The solution of the wave equation in the air beyond the cell consists of superpositions of plane waves of the form $E(x, t) = E(k)e^{i(k \cdot x - \omega t)}$ where $k = n_{\text{air}} k$. When $\theta = 0$ the system has a discrete translation symmetry in $x$, with period $\Lambda$. One may write $E(x + \Lambda \xi, t) = E(x, t)$ so that the free solution may be written as a Fourier series in $x$ as [4]

$$E(x, z, t) = \sum_{l=-\infty}^{\infty} E_l e^{i[(l \Lambda + z - d) / \sqrt{k^2 - (l \Omega)^2} - \omega t]}.$$  (4-40)

Each $l$ corresponds to a diffraction order. To determine the $E_l$, one uses the boundary condition that the tangential $E$ is continuous on the boundary. Ignoring reflections and evaluating Eq 4-40 on the surface $z = d$ allows one to write the $E_l$ as the spatial Fourier transform

$$E_l = \frac{1}{\Lambda} \int_0^{\Lambda} E(x, d, t) e^{-i(l \Omega x)} dx.$$  (4-41)

When $\theta \neq 0$ the system is no longer strictly periodic in $x$. Consider what happens to a ray which enters the bottom of the cell at $x$. Write the local electric field as $E_0 e^{i\phi_0}$ where we are free to choose the initial phase, $\phi_0$. The ray is subjected to some complicated index of refraction which is a function of $x$ and $z$. It bends and exits the cell. Now consider a ray which enters the cell at $x_0 + \Lambda$. Due to the additional distance travelled through the air before hitting the cell, the local phase is $\phi_0 + \tilde{\kappa} \Lambda \sin \theta$ and the local field is $E_0 e^{i(\phi_0 + \tilde{\kappa} \Lambda \sin \theta)}$. Since the cell is periodic in $\Lambda$, the second
ray is subjected to the same complicated index of refraction as the first. It is clear that the trajectory of the ray can not depend on the phase of the electric field where the ray enters the cell since our choice of the origin of $\phi_0$ was completely arbitrary. Therefore one is led to the conclusion that the second ray must follow exactly the same path as the first and that the electric field at each point on the second path must be the same as the first path with the addition of the initial phase difference of $\bar{k}\Lambda \sin \theta$. This can be summarized by the statement that the system possesses the symmetry $E(x + \Lambda \xi, t) = e^{i\bar{k}\Lambda \sin \theta} E(x, t)$ everywhere in space. Define a new field $E'(x, t) = e^{-i\bar{k}x \sin \theta} E(x, t)$. $E'(x, t)$ has the property that it is exactly periodic

$$E'(x + \Lambda \xi) = e^{-i\bar{k}x \sin \theta(x+\Lambda)} E(x + \Lambda \xi)$$
$$= e^{-i\bar{k}x \sin \theta(x+\Lambda)} e^{i\bar{k}\Lambda \sin \theta} E(x)$$
$$= e^{-i\bar{k}x \sin \theta} E(x)$$
$$= E'(x).$$ (4-42)

As before, it can be written as a Fourier series in $x$. Thus the general form for the electric field, consistent with the symmetry, is

$$E(x, z, t) = \sum_{l=-\infty}^{\infty} E_l e^{i[(\bar{k}\sin \theta + l\bar{K})x + (z-d)\sqrt{\bar{k}^2-(\bar{k}\sin \theta + l\bar{K})^2} - \omega t]}.$$ (4-43)

Similarly one can compute the $E_l$ by

$$E_l = \frac{1}{\Lambda} \int_{0}^{\Lambda} E(x, d, t) e^{-i[(\bar{k}\sin \theta + l\bar{K})x - \omega t]} dx.$$ (4-44)
4.3.4 Application to Thermal Convection

In this section I will evaluate the expressions for the index model appropriate to weak thermal convection rolls with the illuminating beam normally incident on the cell. I will, for now, ignore the linear conduction gradient. The index model is

\[
a(x) = \cos(Kx) \\
b(z) = \sin(K'z) \\
n_2 = 0 \\
\theta = 0.
\]

(Eq. 4-45)

Evaluating Eq 4-10 gives

\[
x_1(z) = \frac{K}{K'^2 n_0} \sin(Kx_0) [\sin(K'z) - K'z].
\]

(Eq. 4-46)

The eikonal is then given by Eq 4-38 as

\[
\phi(x_0) = n_0 d + \frac{2}{\pi} n_1 d \cos(Kx_0) + \frac{K^2}{K'^2} \frac{n_1^2}{n_0} d \sin(Kx_0)^2 + O(n_1^3).
\]

(Eq. 4-47)

Notice that the eikonal is given as a function of \( x_0 \). To propagate the solution into the free space above the cell, we must match boundary conditions on the top surface. The free solutions will be expressed as functions of \( x_1 \). Therefore one must re-express Eq 4-47 as a function of \( x_1 \) by inverting the ray path in Eq 4-4 to compute \( x_0 \) in terms of \( x_1 \). See APPENDIX G Inversion of the Ray Path on page 287 for details on this procedure. One finds
\[ x_0 = \pi \frac{n_1 K}{n_0 K^2} \sin(Kx_1) \]
\[ + \frac{n_1^2}{n_0^2} \frac{K}{8K^4} (5K^2\pi^2 - 8K^2 - 16K^2) \sin(2Kx_1) \]
\[ + O(n_1^3). \]  

(4-48)

Insert Eq 4-48 into Eq 4-47 to get the eikonal as a function of \( x_1 \)

\[ \phi(x_1) = d\rho_0 + \frac{2}{\pi} d\rho_1 \cos(Kx_1) \]
\[ - \frac{3}{2\pi} \frac{K^2 n_1^2}{K^2 n_0} \sin(Kx_1)^2 + O(n_1^3). \]  

(4-49)

Similarly, one may compute the amplitude with Eq 4-39 and Eq 4-49

\[ E(x_1) = \sqrt{\frac{\partial x_0}{\partial x_1}} \]
\[ = 1 + \frac{\pi K^2 n_1}{2 K^2 n_0} \cos(Kx_1) \]
\[ + \frac{K^2}{16K^4} [(9K^2\pi^2 - 32K^2 - 16K^2) \cos(2Kx_1) \]
\[ - K^2\pi^2 \frac{n_1^2}{n_0^2} + O(n_1^3). \]  

(4-50)

4.3.5 Propagation of the Electric Field Beyond the Cell for Thermal Convection

The electric field inside on the top surface of the cell is now given by

Eq 4-27. In order to match boundary conditions we need the solution in

the form of Eq 4-40 by computing the Fourier transform in \( x \). Since the

system is periodic in \( x \), one may write the amplitude and phase gratings as

Taylor series. The scalar electric field can then be written as

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\[ E(x) = E_0 (a_0 + a_1 \cos(Kx) + a_2 \cos(2Kx)) \times e^{iK(\phi_0 + \phi_1 \cos(Kx) + \phi_2 \cos(2Kx)) - i\omega t}. \]  

This form can then be matched to the free solutions. See APPENDIX H Fourier Transform of the Eikonal Solution on page 289. One finds the coefficients \( a_0 \), \( a_1 \), \( a_2 \), \( \phi_1 \), and \( \phi_2 \) in Eq 4-51 as

\[
\begin{align*}
    a_0 &= 1 - \frac{\pi^2}{16} \frac{K^4}{K'^4} \frac{n_1^2}{n_0^2} \\
    a_1 &= \frac{\pi}{2} \frac{K^2}{K'^2} \frac{n_1}{n_0} \\
    a_2 &= \frac{K^4}{K'^4} \left( \frac{9}{16} \pi^2 - 2 - \frac{K'^2}{K^2} \right) \frac{n_1^2}{n_0^2} \\
    k\phi_1 &= 2 \frac{k}{K} n_1 \\
    k\phi_2 &= \frac{3\pi}{4} \frac{k}{K'} \frac{K^2}{K'^2} \frac{n_1^2}{n_0} = \frac{3\pi}{8} \frac{K^2}{K'^2} \frac{n_1}{n_0} \left( k\phi_1 \right).
\end{align*}
\]  

Consider the magnitudes of the various terms in Eq 4-52. The magnitude of the amplitude grating terms, \( a_1 \) and \( a_2 \), should be compared to unity and the phase grating terms, \( k\phi_1 \) and \( k\phi_2 \), should be compared to \( 2\pi \). One can identify three limits. When \( k\phi_2 \leq 2\pi \), one must retain the \( a_1 \) term, but since \( K/K' \approx 1 \) and \( n_1/n_0 \ll 1 \), \( a_2 \) can always be ignored. Similarly \( a_0 \) can be taken to be unity. I will call this the strongly diffracting limit. When \( k\phi_2 \gg 2\pi \), which occurs when \( n_1/n_0 \ll \sqrt{\lambda/\Lambda} \), one may drop all terms except for \( k\phi_1 \) and \( a_1 \). In this limit, the electric field beyond the cell takes the simple form
\[ E(x, z) = e^{i(k_0 x - \omega t)} \sum_{l=-\infty}^{\infty} E_l e^{i [lKx + (z - d) \sqrt{k^2 - (lK)^2}]} , \quad (4-53) \]

where

\[ E_l = E_0 e^{il\pi/2} \left[ J_l(k\phi_1) + i \frac{a}{2} [J_{l+1}(k\phi_1) - J_{l-1}(k\phi_1)] \right] . \quad (4-54) \]

Lastly when \( k\phi_1 \ll 2\pi \), which occurs when \( n_1/n_0 \ll L/\lambda \), this sum is dominated by the \( l = 0 \) and \( l = \pm 1 \) terms since \( J_l(x) \approx \frac{1}{l!} \left( \frac{x}{2} \right)^l + O(x^2) \) for \( l \geq 0 \) and \( J_{-l}(x) = (-1)^l J_l(x) \) for \( l < 0 \). I will call this the weakly diffracting limit. Expand the Bessel functions and square to get the intensity as

\[ \frac{I(x, z)}{I_0} = 1 + 4 \frac{k}{K} n_1 \cos(Kx) \sin\left[ \frac{K^2}{2k} (z - z_0) \right] + O(n_1^2) , \quad (4-55) \]

where \( z_0 \) is the same as defined in Eq 4-17. The intensity of the beam passing through the sample cell when one sets \( n_1 \to 0 \) is denoted \( I_0 \). In the laboratory, this is easily measured by reducing the temperature difference across the cell, until the convection vanishes and the cell is in the conduction state. It is useful to compare this to the corresponding result from geometrical optics. This is simplest if one restricts the comparison to the intensity along the line \( x = 0 \) where the intensity is maximal. This was computed in Eq 4-21. For viewing planes near the cell where \( z \) is much less than the geometrical focal length, one finds the intensity grows linearly with distance from the cell as

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\[
\frac{I(0, z)}{I_0} = 1 + 2 \frac{n_1}{n_{\text{air}}} K^2 \left(z - z_0\right),
\]

which agrees with the corresponding limit of Eq 4-55. This is illustrated in Figure 4-9.

![Graph showing intensity as a function of distance from the cell](image)

**FIGURE 4-9.** Intensity as a Function of Distance from the Cell.

This figure shows the intensity along the line \( x = 0 \) for typical parameters. Here I have used \( n_1/n_0 = 2.5 \times 10^{-5} \), \( \lambda = .5 \mu \text{m} \), \( d = .5 \text{mm} \), and \( \Lambda = 1 \text{ mm} \). This is in the weakly diffracting limit, \( k\phi = .01 \).

Since \( K \) scales as \( d^{-1} \), one expects the maximum of the shadowgraph signal, \( I/I_0 - 1 \), to be proportional to the thickness of the cell, suggesting that thick cells would give a more sensitive measurement of the amplitude of the index modulation. This is somewhat misleading. This would be true if one operated the experiment at constant \( n_1 \). This is rarely the case. One usually chooses to run an experiment in a specific \( \varepsilon \) range. The
critical temperature differences scales as $d^3$ and $n_1$ is proportional to $\Delta T - \Delta T_c$. Therefore Eq 4-55 predicts that in the weakly diffracting limit the shadowgraph sensitivity actually scales as $d^{-2}$. If one is interested in phenomena near onset, there is a strong incentive to measure very thin cells. This is largely responsible for the success of experiments in gas convection, where large cells with thicknesses in the range of 0.1 mm are common [5].

4.3.6 Development of Caustic-like Solutions

One of the deficiencies of the geometrical optics treatment of the shadowgraph problem is the existence of caustics and spatial discontinuities of the intensity. One expects that physical optics should reproduce caustic-like solutions in the limit of $\lambda/n_1 d \to 0$ where the two solutions must approach each other. This is shown in Figure 4-10. One striking difference between the solutions is that physical optics predicts light and dark bands beyond the maximum peak. As one makes $\lambda/n_1 d$ smaller, the number and amplitude of these peaks increases and their spacing decreases.

4.3.7 The Effect of the Conduction Temperature Profile

Consider the effect of the linear gradient caused by the conduction temperature profile; $n_2 \neq 0$ in Eq 4-1. Again only consider normal incidence so that $x_0(0) = x_0$. Equation 4-9 becomes
FIGURE 4-10. Development of Caustic-like Solution

This figure shows the intensity along the line $x = 0$ for the same parameters as in Figure 4-9 except $n_1/n_0 = 2.5 \times 10^{-3}$, so that $k_0 = 1$. The dashed line is the geometrical optics solution showing the caustic. The solid line is the physical optics result. The intensity becomes large but does not diverge.

\[ K \sin(Kx_0) \sin(K'z) + n_2x_1'(z) + (n_0 + n_2K'z)x_1''(z) = 0. \quad (4-57) \]

This has the solution

\[
x_1(z) = \frac{K}{K'^2n_2} \sin(Kx_0) \left\{ \ln \left( \frac{n_0}{n_0 + n_2K'z} \right) + \cos \left( \frac{n_0}{n_2} \right) [\text{ci} \left( \frac{n_0}{n_2} \right) - \text{ci} \left( \frac{n_0}{n_2} + K'z \right)] \\
+ \sin \left( \frac{n_0}{n_2} \right) \left[ \text{si} \left( \frac{n_0}{n_2} \right) + \text{si} \left( \frac{n_0}{n_2} + K'z \right) \right] \right\}. \quad (4-58)
\]

The \text{ci}(x) and \text{si}(x) functions are the standard cosine integral and sine integral special functions [6]. Although this result is exact, it is useful to
expand it for small values of $n_2$. This must be done with some care, since Eq 4-58 contains terms such as $\cos(n_0/n_2)$ which have essential singularities as $n_2 \rightarrow 0$. One must first expand out the $ci$ and $si$ using the asymptotic expansion

\[
\begin{align*}
\text{si}(z) & \sim \frac{\pi}{2} - \frac{\cos z}{z} \\
\text{ci}(z) & \sim \frac{\sin z}{z},
\end{align*}
\]  

(4-59)

for large values of $z$. With these expansions the essential singularities cancel and one finds

\[
x_1(z) = \frac{K}{K'^2 n_0} \sin(K x_0) [\sin(K'z) - K'z] + \frac{K n_2}{K'^2 n_0^2} \left[ 1 + \frac{1}{2} K'^2 z^2 - \cos(K'z) - K'z \sin(K'z) \right] + O(n_2^2).
\]  

(4-60)

Inserting Eq 4-60 into Eq 4-38 and Eq 4-39 yields the amplitude and phase grating coefficients $a_1$, $\phi_1$, and $\phi_2$ in Eq 4-51 as

\[
\begin{align*}
a_0 &= 1 - \frac{\pi^2}{16} \frac{K^4}{K'^4} \frac{n_1^2}{n_0^2} \left[ 1 - \left( 1 + \frac{4}{\pi^2} \right) \frac{n_2}{n_0} \right] \\
a_1 &= \frac{\pi}{2} \frac{K^2}{K'^2} \frac{n_1}{n_0} \left[ 1 - \frac{1}{2} \left( 1 + \frac{4}{\pi^2} \right) \frac{n_2}{n_0} \right] \\
k\phi_1 &= 2 \frac{k}{K'} n_1 \\
k\phi_2 &= \frac{3\pi}{4} \frac{k}{K'} \frac{K^2}{K'^2} \frac{n_1^2}{n_0} \left[ 1 - \frac{1}{2} \left( 1 + \frac{16}{3 \pi^2} \right) \frac{n_2}{n_0} \right].
\end{align*}
\]  

(4-61)
4.3.8 The Effect of a Finite Size Light Source

Real shadowgraph apparatus generate the incident beam of light by placing a small light source at the focus of an objective lens. This light source has a finite extent so the resulting beam is not truly parallel. One can analyze this effect by using Huygen's construction. Each point on the light source generates an incident beam at a small angle from the normal as illustrated in Figure 4-11. Each beam creates a shadowgraph image displaced in space from the others. I will assume that the light source is approximately monochromatic but sufficiently temporally incoherent that one must sum the shadowgraph images from each point on the light source. This results in a smeared image. To compute the extent of the smearing, one must first compute the shadowgraph image which results from a beam at an angle to the normal. Consider the index model in Eq 4-45 but with a finite $\theta$. Compute the ray path as before. The solution to Eq 4-8 becomes

$$x_0(z) = x_0 + z\bar{\theta}, \quad (4-62)$$

where $\bar{\theta}$ is the angle the ray makes inside the medium as corrected by Snell's law, $n_{\text{air}}\sin \theta = n_0 \sin \bar{\theta}$, and the approximation $\tan \bar{\theta} \approx \bar{\theta}$ has been used since $\bar{\theta}$ is small. Similarly
The light source is modeled as a uniformly illuminated disk, of radius \( r \), positioned at the focal point \( f \), of an objective lens. The diagram shows the light emitted from a point on the edge of the source, creating a plane parallel beam at an angle \( \theta \).

\[
x_1(z) = \frac{1}{n_0} \frac{K}{K'}^2 \sin(Kx_0)(\sin(K'z) - K'z) \\
+ \frac{2\theta}{n_0} \frac{K^2}{K'}^2 \cos(Kx_0) \sin(K'z/2) \\
\times \left\{ K'z \cos(K'z/2) - \left[ 2 - \frac{K^2}{K'}^2 \right] \sin(K'z/2) \right\}.
\]

The eikonal is then given by Eq 4-37 as
\[
\phi(x_1) = \sin \theta + dn_0 + dn_1 \left[ \frac{2}{\pi} \cos(Kx_1) + \frac{K}{K'} \bar{\theta} \sin(Kx_1) \right] + O(n_1^2).
\]

The amplitude grating is
\[
E(x_1) = 1 + \frac{n_1}{n_0} \left[ \frac{\pi K^2}{2 K''} \cos(Kx_1) - \frac{K}{K'} \left( \frac{4K^2}{K''^2} + 2 \right) \bar{\theta} \sin(Kx_1) \right] + O(n_1^2).
\]

These expressions can now be used to compute the \( E_I \). Comparing Eq 4-64 to Eq 4-44 and Eq 4-65 to Eq 4-45 shows that the corrections due to the finite incident angle are all of order \( dn_1 \bar{\theta} \). As long as this is small, one may neglect the \( \theta \) dependence in both the amplitude and phase gratings. It only survives in propagation of the beam beyond the cell. The shadow-graph signal is simply shifted in space. The solution is therefore simply Eq 4-43 with the same \( E_I \) as the normal incidence. Therefore one finds that, to a good approximation,
\[
I(x, z, \theta) = I(x + z \tan\theta, z, 0),
\]

where \( I(x, z, 0) \) is given by Eq 4-48. This result is for the incident beam rotated about the \( y \) axis. For the beam rotated about the \( x \) axis, the image is shifted in the \( y \) direction. Since the rolls have no variation in the \( y \) direction, there is no smearing. The effect of the finite aperture can now be evaluated by integrating this expression over the angular range spanned by the aperture. I am assuming that the light source has a uniform inten-
sity over its surface. One integrates over the range of angles defined in Figure 4-12.

\[
\frac{I(x, z, \theta_a)}{I_0} = \frac{1}{\pi \sigma_a^2} \int_{-\theta_a}^{\theta_a} d\theta_y \int_{-\theta_x}^{\theta_x} d\theta_x I(\theta_x, \theta_y, x, z)
\]

\[
= \frac{1}{\pi \sigma_a^2} \int_{-\theta_x}^{\theta_x} d\theta_x \sqrt{\sigma_a^2 - \theta_x^2} \times \left\{ 1 + 4 \frac{k}{K^2} n_1 \cos(K(x - z \tan \theta_x)) \sin \left( \frac{zK^2}{2k} \right) \right\}
\]

(4-67)

\[
\approx 1 + 4 \frac{k}{\pi \sigma_a^2} \frac{k}{K^2} n_1 \cos(Kx) \sin \left( \frac{zK^2}{2k} \right) \times \int_{-\theta_x}^{\theta_x} d\theta_x \sqrt{\sigma_a^2 - \theta_x^2} \cos(Kz \theta_x)
\]

\[
= 1 + 4 \frac{k}{K^2} n_1 \cos(Kx) \sin \left( \frac{zK^2}{2k} \right) \frac{J_1(Kz \sigma_a)}{Kz \sigma_a/2}.
\]
The final result is identical to Eq 4–48 except that the intensity modulation is modified by the factor \( \frac{I_1(K_x \theta_e)}{K_x \theta_e/2} \) which approaches unity for \( \theta_e \to 0 \).

4.3.9 Shadowgraph Signal from White Light Sources

Early work on convection often used white light sources owing to their brightness and ready availability. Modern instruments use LED's which while much more monochromatic than incandescent light sources, still have an appreciable spectral width. This will affect the shadowgraph signal in two ways. The dispersion in the medium will cause \( n_0 \) and \( n_1 \) to be functions of frequency. This causes a chromatic aberration in the focusing from the weak convective lenses. Additionally, even in the absence of chromatic aberration, the shadowgraph intensity has an explicit wavelength dependence. One again I will concentrate on the weakly diffracting limit where there is a simple analytic form. Assume a simple model of a light source with constant intensity over a band of frequencies \( \Delta \lambda \), centered about \( \lambda \). One can integrate Eq 4–55 to get

\[
\frac{I(x, z)}{I_0} = \frac{1}{\Delta \lambda} \int_{\lambda-\Delta \lambda/2}^{\lambda+\Delta \lambda/2} \left( 1 + 8n_1 d \cos(Kx) \frac{\sin\left[\frac{\pi(z-z_0)}{\lambda n_{air}^2}\right]}{\lambda} \right) d\lambda
\]

\[
= 1 + \frac{8n_1 d}{\Delta \lambda} \cos(Kx) \left\{ \sin\left[\frac{\pi(z-z_0)}{\lambda^2 n_{air}^2}\right] (\lambda + \Delta \lambda) \right. \\
\left. - \sin\left[\frac{\pi(z-z_0)}{\lambda^2 n_{air}^2}\right] (\lambda - \Delta \lambda) \right\}.
\]
When $\Delta \lambda \approx \lambda$, this can be simplified. Introduce $\kappa = K^2 (z - z_0)/2k$ so that Eq 4-68 becomes

$$\frac{I(x, z)}{I_0} = 1 + 4n_1 \frac{k}{K} \cos(Kx) \sin(\kappa) \times \left[ 1 - \left( \frac{\Delta \lambda}{\lambda} \right)^2 \left( \frac{2 - \kappa^2 - 2\kappa \cot \kappa}{24} \right) \right]. \quad (4-69)$$

This result should be evaluated near the maximum intensity where $\kappa \approx \pi/2$. The conclusion is that the pure monochromatic result of Eq 4-55 is only reduced in amplitude by order $\left( \frac{4\lambda}{\lambda} \right)^2$. This explains why acceptable images can be obtained even with broadband light sources.
4.3.10 Superposition of Patterns

So far, only one dimensional periodic patterns have been considered. Many cells have two dimensional patterns so it is useful to consider the shadowgraph generated by superpositions of periodic patterns. This section will generalize the derivation of the ray path given in Section 4.2.1 on page 189. I will neglect the linear conduction gradient and take \( n_2 = 0 \). I will also only consider the case of the incident beam normal to the cell. Moreover, the analysis will only be carried to \( O(n_1^2) \). The generalized index model is now

\[
n(x, y, z) = n_0 + n_1 a(x, y)b(z). \tag{4-70}
\]

The ray path is a vector function \( x(z) = (x(z), y(z)) \), parametrized by \( z \) so that one has the coupled Euler-Lagrange equations

\[
\frac{\partial f}{\partial x} - dx \left( \frac{\partial f}{\partial x'} \right) = 0
\]

\[
\frac{\partial f}{\partial y} - dy \left( \frac{\partial f}{\partial y'} \right) = 0, \tag{4-71}
\]

where \( f(x, x', y, y', z) = n(x, y, z)\sqrt{1 + x'^2 + y'^2} \). As before, I will seek a solution as a series in the small parameter \( n_1 \). Expand out both \( x(z) \) and \( y(z) \) to \( O(n_1^2) \) as

\[
x(z) = x_0(z) + x_1(z)n_1 + O(n_1^2)
\]

\[
y(z) = y_0(z) + y_1(z)n_1 + O(n_1^2). \tag{4-72}
\]
Insert Eq 4-72 into Eq 4-71 and collect the $O(1)$ terms. One finds immediately that $x_0''(z) = y_0''(z) = 0$. Satisfying the boundary conditions gives the trivial result $(x_0(z), y_0(z)) = (x_0, y_0)$. Similarly the $O(n_1)$ equations are decoupled

$$b(z) \frac{\partial}{\partial x} a(x, y) - n_0 x_1''(z) = 0$$
$$b(z) \frac{\partial}{\partial y} a(x, y) - n_0 y_1''(z) = 0. \quad (4-73)$$

The $O(n_1^2)$ equations are not decoupled which explains why this analysis will not be carried further. Solving Eq 4-73 yields the ray path

$$x(z) = x_0 + n_1 \nabla_2 a(x, y) h(z), \quad (4-74)$$

where $x_0 = (x_0, y_0)$, $x = (x, y)$, $\nabla_2 = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ and

$$h(z) = \frac{1}{n_0} \int_0^z (z' - z) b(z') dz'. \quad (4-75)$$

Given the ray path, one can now compute the eikonal and the amplitude grating. The eikonal is computed by inserting Eq 4-74 into Eq 4-32. One finds

$$\phi(x) = \int_0^d n(x(z), y(z), z) dz$$
$$= \int_0^d \left[ n_0 + n_1 a(x_0, y_0) b(z) + O(n_1^2) \right] dz$$
$$= n_0 d + n_1 a(x_0, y_0) d\langle b(z) \rangle + O(n_1^2). \quad (4-76)$$
Similarly the amplitude grating is given by inserting Eq 4-74 into Eq 4-36. One must compute the change in an area element as it follows the ray path from the bottom of the cell to the top. To do this one must compute the Jacobian derivative

$$E(x) = E_0 \left| \begin{array}{cc} \frac{\partial x(d)}{\partial x_0} & \frac{\partial x(d)}{\partial y_0} \\ \frac{\partial y(d)}{\partial x_0} & \frac{\partial y(d)}{\partial y_0} \end{array} \right|^\frac{1}{2}$$

$$= E_0 \left[ 1 + n_1 h(d) \frac{\partial^2}{\partial x^2} a(x) + n_1 h(d) \frac{\partial^2}{\partial x \partial y} a(x) \right]^{\frac{1}{2}}$$

$$= E_0 \left[ 1 + \frac{n_1}{2} h(d) \nabla^2 a(x) \right] + O(n_1^2).$$

Equation 4-76 and Eq 4-77 shows that both the amplitude and eikonal are linear functions of $a(x)$. One can use the linearity and the fact that this expansion is of $O(n_1^2)$ to write the electric field on the top of the cell in a particularly simple form. Consider the case where $a(x)$ can be written as the sum of two terms for which the amplitude and phase grating terms have been individually computed, $a(x) = a_1(x) + a_2(x)$. One example is two superimposed roll patterns oriented in different directions. Let the $O(n_1)$ amplitude and phase grating terms for $a_1(x)$ alone, be $A_1$ and $p_1$, so that one may write the electric field, associated with $a_1(x)$, on the top surface as

$$E(x) = E_0 e^{i(k_0 d - \omega t)} (1 + n_1 A_1(x)) e^{i k_1 p_1(x)}$$

$$= E_0 e^{i(k_0 d - \omega t)} E_{1h}(x).$$
Here $E_1^{sh}(x)$ represents the portion of the electric field which gives rise to the shadowgraph signal. One has a similar expression for $a_2(x)$. One may now use the linearity of Eq 4-76 and Eq 4-77 to write the electric field for the combined system as

$$E(x) = E_0 (1 + n_1 A_1(x) + n_1 A_2(x)) e^{i k (n_0 d + n_1 p_1(x) + n_1 p_2(x)) - i \omega t} + O(n_1^2)$$

$$= E_0 e^{i k n_0 d - i \omega t} (1 + n_1 A_1(x)) e^{i n_1 p_1(x)} \times$$

$$(1 + n_1 A_2(x)) e^{i n_1 p_2(x)} + O(n_1^2)$$

$$= E_0 e^{i k n_0 d - i \omega t} E_1^{sh}(x) E_2^{sh}(x) + O(n_1^2).$$  \hspace{1cm} (4-79)

This implies

$$E^{sh}(x) = E_1^{sh}(x) E_2^{sh}(x).$$  \hspace{1cm} (4-80)

One is led to the strikingly simple result that the electric field on the top surface of the cell is to $O(n_1^2)$ simply the product of the electric fields of the component rolls! Since the problem of a shadowgraph from a single roll has previously been solved, one can construct the solution to a superposition of rolls from the single roll solutions. To illustrate the procedure, consider the problem of the shadowgraph which results from a square or rhombic lattice. The model is

$$a(x) = \cos(K_1 \cdot x) + \cos(K_2 \cdot x),$$  \hspace{1cm} (4-81)

where $|K_1| = |K_2|$. The electric field on the top of the cell is therefore
\[ E^\text{th}(x) = \left( \sum_{l=-\infty}^{\infty} E_l e^{i lK_1 \cdot x} \right) \left( \sum_{m=-\infty}^{\infty} E_m e^{i mK_2 \cdot x} \right) \]

\[ = \sum_{l=-\infty}^{\infty} E_l E_m e^{i (lK_1 + mK_2) \cdot x}. \]  

(4-82)

\( E_l \) is the single roll coefficient computed in Eq 4-54. The free solution above the cell is therefore

\[ E(x) = e^{i(k_0 d - \omega x)} \sum_{l=-\infty}^{\infty} E_l E_m e^{i (lK_1 + mK_2) \cdot x - i \sqrt{k^2 - (lK_1 + mK_2)^2} z}. \]  

(4-83)

One can square this to get the intensity as

\[ I(x) = \sum_{l=-\infty}^{\infty} E_l E_{l'}^* E_m E_{m'}^* e^{i [(l-l')K_1 + (m-m')K_2] \cdot x} \times \]

\[ e^{i \left( \sqrt{k^2 - (lK_1 + mK_2)^2} - \sqrt{k^2 - (l'K_1 + m'K_2)^2} \right) z}. \]  

(4-84)

This result can be generalized to three rolls as

\[ I(x) = \sum_{l=-\infty}^{\infty} E_l E_{l'}^* E_m E_{m'}^* E_n E_{n'}^* e^{i [(l-l')K_1 + (m-m')K_2 + (n-n')K_3] \cdot x} \times \]

\[ e^{i \left( \sqrt{k^2 - (lK_1 + mK_2 + nK_3)^2} - \sqrt{k^2 - (l'K_1 + m'K_2 + n'K_3)^2} \right) z}. \]  

(4-85)

The generalization to an arbitrary number of rolls is clear. This result is illustrated for both square and hexagonal patterns in Figure 4-13.

Equation 4-84 may be further simplified for two specific cases. First rewrite the exponent by using the fact that \( \bar{k} \ll K \) as

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FIGURE 4-13. Computed Shadowgraph Images of Squares and Hexagons

All images are computed for d=0.5 mm, λ=1 mm, λ=0.5 μm and are evaluated on the plane which gives the maximum intensity modulation. The left images are evaluated in the weakly diffracting limit and have \( n_1/n_0 = 2.5 \times 10^{-6} \) so that \( k\phi = 0.01 \). They are computed for a plane 1 m from the cell. The gray scale for each image is set so that the maximum intensity is white and the minimum intensity is black. The right images approach the geometrical optics limit. They have \( n_1/n_0 = 2.5 \times 10^{-4} \) so that \( k\phi = 1 \) and are computed for a plane 0.46 m from the cell. The peak intensity in the images are 1.04 and 1.06 for the weak squares and hexagons and 8.5 and 24.7 for the strong squares and hexagons respectively. The peaks in the right images are the physical optics equivalents of caustics.
\[ I(x) = \sum_{l, l'} \sum^{\infty}_{m, m'} E_l E_{l'}^* E_m E_{m'}^* e^{i((l-l')K_1 + (m-m')K_2) \cdot x} \times \]
\[ e^{i \frac{K_2^2}{2}(r^2 - l^2 + m'^2 - m^2 + 2K_1 \cdot K_2 (l'm' - lm))} \]

Consider the case of a square lattice where \( K_1 \cdot K_2 = 0 \). One may then write

\[ I_{sq}(x) = \left( \sum_{l, l'} \sum^{\infty}_{m, m'} E_l E_{l'}^* e^{i((l-l')K_1 \cdot x + i \frac{K_2^2}{2}(r^2 - l^2))} \right) \times \]
\[ \left( \sum_{m, m'} \sum^{\infty}_{m, m'} E_m E_{m'}^* e^{i(m-m')K_2 \cdot x + i \frac{K_2^2}{2}(m'^2 - m^2))} \right) \] (4-87)

\[ = I_{roll}(K_1 \cdot x)I_{roll}(K_2 \cdot x). \]

The shadowgraph for the square lattice is simply the product of the shadowgraphs of the individual rolls. The second case is the weakly diffracting limit where it has been shown that the single roll shadowgraph intensity pattern is dominated by \( l = 0 \) and \( l = \pm 1 \) as in Eq 4-55. Analogously, in the weakly diffracting limit, Eq 4-86 is dominated by \( lm = 0 \) and \( lm = \pm 1 \). Therefore the \( 2K_1 \cdot K_2 (l'm' - lm) \) term again vanishes and one is again left with the product form of Eq 4-87. In the weakly diffracting limit, one may generalize Eq 4-87 to a superposition of an arbitrary number of discrete rolls. One finds

\[ I(x) = \prod_{i=0}^{\infty} I_{roll}(K_1 \cdot x). \] (4-88)
One may take this result to the continuous limit by first taking the logarithm of both sides so that one has

\[ \ln I(x) = \sum_{i=0}^{\infty} \ln I_{\text{roll}}(K_i \cdot x). \]  \hspace{1cm} (4-89)

The final result for the weakly diffracting limit is

\[ I(x) = e^{\int d^2 K \ln I_{\text{roll}}(K \cdot x)}. \]  \hspace{1cm} (4-90)
4.4 Shadowgraph of Electrohydrodynamic Convection of Nematic Liquid Crystals

Liquid crystals have an anisotropic dielectric constant which is aligned with the local director. The effect of electroconvection is to spatially modulate the orientation of the optical axis as shown in Figure 4-14. The local director angle is $\Theta(x)$. This is in contrast to thermal convection where the focusing is due to local changes in the magnitude of the index of refraction. Uniaxial crystals have an index of refraction parallel to the optical axis which is different than the index of refraction along the two axes perpendicular to the optical axis. These are known as the extraordinary and ordinary indices. Beams travelling at an angle to the optical axis have an effective index of $n_{\text{eff}} = n_o$ when the polarization is along y and

![Figure 4-14. Experimental Setup for Convection in Nematic Liquid Crystals](image)

Only the $x$ polarization is considered. The $y$ polarization passes through the cell undiffracted.
\[ n_{\text{eff}} = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \gamma + n_e^2 \cos^2 \gamma}}, \]  

(4-91)

when the polarization is in the plane. Here \( \gamma \) is the angle between the director, which forms the optical axis, and the \( \mathbf{k} \) vector of the light ray. (See APPENDIX I Effective Index of Refraction for Uniaxial Crystals on page 291.) Electroconvection rotates the director in the \( x-z \) plane changing the local value of \( \gamma \). Therefore \( y \) polarized waves are subjected to a constant effective index of refraction and do not bend as they propagate through the cell. For this reason, most liquid crystal shadowgraph apparatus illuminate the cell with \( x \) polarized light to enhance the contrast of the image. The rest of this discussion will be restricted to this case and will only consider light which is incident normal to the bottom plate. Parametrize the ray path as \( x(z) \) so that one may write

\[ \gamma(x) = \frac{\pi}{2} - \tan^{-1}(x'(z)) - \theta(x). \]  

(4-92)

For weak electrohydrodynamic convection the director distortion, to a good approximation, is given by

\[ \theta(x) = \theta_0 \sin(Kx) \sin(K'z), \]  

(4-93)

where \( \theta_0 \) is the amplitude of the director distortion. With this choice of \( \theta(x) \), the system has the same symmetry as the thermal convection case; a mirror reflection symmetry about \( x = 0 \), and a discrete translation symmetry with period \( \Lambda \). Therefore, as before, the electric field must have the
form in Eq 4-43 and in APPENDIX H Fourier Transform of the Eikonal Solution on page 289. One may again use Fermat's principle of least action to compute the ray path

\[ \frac{\partial f}{\partial x} - \frac{d}{dx} \left( \frac{\partial f}{\partial x'} \right) = 0, \]  

(4-94)

where \( f(x, x', z) = n(x, x', z) \sqrt{1 + x'^2} \) which is similar to Eq 4-3 except that now the index of refraction is a function of \( x, x' \), and \( z \). Look for a solution as a series expansion in the small parameter, which in this case is \( \theta_0 \).

Write the ray path as

\[ x(z) = x_0(z) + x_1(z) \theta_0 + x_2(z) \theta_0^2 + \mathcal{O}(\theta_0^3). \]  

(4-95)

Insert this ansatz into Eq 4-94 and match the orders of \( \theta_0 \) to get a series of ODE's for the \( x_i(z) \). The \( \mathcal{O}(1) \) equation is the rather complicated expression

\[ x_0''(z) \left[ n_e^2 n_o^2 - n_e^4 - n_o^4 + (n_e^4 - n_o^4) \cos \{ 2 \tan^{-1} [x_0'(z)] \} \right] = 0. \]  

(4-96)

Fortunately, the left hand side of this ODE is proportional to \( x_0''(z) \) so it is solved by a simple linear function

\[ x_0(z) = A z + x_0. \]  

(4-97)

I will only consider the case of normal incidence so that solution matching the boundary conditions at the \( z = 0 \) implies \( x_0''(z) \) is simply the
constant function \( x_0(z) = x_0 \). Insert this back into Eq 4-94 to get the \( O(\theta_0) \) ODE

\[
x_1''(z) + K' \left( \frac{n_e^2 - n_o^2}{n_e^2 - 2n_o^2} \right) \sin(Kx_0) \cos(K'z) = 0.
\]

(4-98)

The solution which satisfies the boundary conditions is

\[
x_1(z) = -\frac{2}{K'} \left( \frac{n_e^2 - n_o^2}{n_e^2 - 2n_o^2} \right) \sin(Kx_0) \sin^2 \left( \frac{K'z}{2} \right).
\]

(4-99)

One can now compute the lowest nonzero contributions to the eikonal and the amplitude. From Eq 4-32 one has

\[
\phi(x_1) = n_e d \left[ 1 - \frac{n_o^2(n_e^2 - n_o^2)}{4(n_e^2 - 2n_o^2)^2} \sin^2(Kx_1) \theta_0^2 + O(\theta_0^3) \right].
\]

(4-100)

Correspondingly one finds the amplitude grating from Eq 4-39 to be

\[
E(x_1) = 1 + \frac{K}{K'} \left( \frac{n_e^2 - n_o^2}{n_e^2 - 2n_o^2} \right) \cos(Kx_1) \theta_0 + O(\theta_0^2).
\]

(4-101)

Match this solution on the top surface to the free solutions in the air (or glass) above the cell. One finds the amplitude and phase grating terms

\[
a_1 = \frac{K}{K'} \left( \frac{n_e^2 - n_o^2}{n_e^2 - 2n_o^2} \right) \theta_0
\]

\[
k \phi_1 = 0
\]

\[
k \phi_2 = \frac{\pi}{8} \frac{k}{K'} \frac{n_o^2(n_e^2 - n_o^2)}{(n_e^2 - 2n_o^2)^2} \theta_0^2.
\]

(4-102)

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As with thermal convection, the amplitude grating terms do not scale with the thickness of the cell. The phase grating term does. Therefore one may not necessarily ignore the phase grating term even though the lowest non-vanishing term is proportional to $\theta_0^2$. In fact, for most experimental cells, $d/\lambda \sim 100$ and $\theta_0 \sim .01$ so that the amplitude and phase grating terms are actually of the same size; $k\phi_2 \sim a_1$. Of course this is a statement of experimental happenstance. One could design cells where this relationship does not hold. The electric field in the air above the cell is of the form:

$$\frac{E(x,z)}{E_0} = \sum_{n=-\infty}^{\infty} E_n e^{i\left(nKx+(z-d)\sqrt{K^2-(nK)^2}\right)/\lambda}$$, \hspace{1cm} (4-103)

where

$$E_n = \begin{cases} 
J_{\frac{n}{2}}(k\phi_2) e^{ikx} & n \text{ even} \\
\frac{a_1}{2} \left[ J_{\frac{n+1}{2}}(k\phi_2) e^{i\left(\frac{n+1}{2}\right)x} + J_{\frac{n-1}{2}}(k\phi_2) e^{i\left(\frac{n-1}{2}\right)x} \right] & n \text{ odd.}
\end{cases}$$ \hspace{1cm} (4-104)

One gets a simplified expression for the intensity in the limit that $k\phi_2 \sim a_1 \ll 1$

$$\frac{I}{I_0} = 1 + 2a_1 \cos(Kx) \cos\left[\frac{K^2(z-d)}{2k}\right] + 2k\phi_2 \cos(2Kx) \sin\left[\frac{2K^2(z-d)}{k}\right].$$ \hspace{1cm} (4-105)

Eq 4-105 should be compared to the corresponding expression for thermal convection in Eq 4-55. One immediate difference is that the liquid
crystal result has a finite intensity on the top surface of the cell. Here the amplitude grating term dominates. In the $x$ direction, it has the same wavelength as the pattern. As one images planes successively further from the cell, this term decays until it vanishes altogether at a distance $z_{\text{vanish}} = n_{\text{air}} \Lambda^2 / (2 \lambda) + d$. Conversely, the phase grating term has a wavelength half that of the pattern and grows from zero on the top surface. It too vanishes at $z_{\text{vanish}}$. This is illustrated schematically in Figure 4-15. Finally, shadowgraph intensity patterns computed as a function of $x$ and $z$ are shown in Figure 4-16, Figure 4-17, and Figure 4-18. They show the results of a cell which progresses from $\theta_0 = 0.001$, where the shadowgraph is dominated by the amplitude grating term to $\theta_0 = 0.1$ where the shadowgraph is dominated by the phase grating term.
FIGURE 4-15. Z dependence of Amplitude and Phase Grating Terms in the Weakly Diffracting Limit.

This figure illustrates the behavior of the amplitude and phase grating terms in Eq 4-105. The abscissa is the (scaled) distance of the viewing plane from the top surface of the cell. The ordinate is the relative contribution of each term. Recall that the x dependence of the amplitude grating has period $\Lambda$ and the phase grating has period $\Lambda/2$. This diagram shows that on the surface of the cell, the intensity pattern is purely from the amplitude grating. At intermediate distances the pattern is a combination of the terms. The pattern vanishes entirely at $z_{\text{vanish}}$. 
This figure shows the intensity of the liquid crystal shadowgraph as a function of x and z. The image represents a hypothetical liquid crystal cell with $d = 25 \, \mu m$, $\Lambda = 50 \, \mu m$, $\lambda = 0.5 \, \mu m$, $n_e = 1.5$, $n_o = 1.6$. This image has $\theta_0 = .001$. For these parameters $z_{\text{vanish}} = 2.5 \, \text{mm}$. The image for these parameters is completely dominated by the amplitude grating term. The image shows intensity modulations with only the same period as the pattern.
FIGURE 4-17. Intensity of Electroconvection Shadowgraph (II)

This figure shows the intensity pattern generated from a cell with the same parameters as in Figure 4-16 but with $\theta_0 = .01$. In this image, the amplitude and phase grating terms are nearly of the same magnitude. On the surface of the cell, the image has the periodicity as the pattern. Further from the cell, the intensity pattern has a substantial component with a spatial period of $\Lambda/2$. 
FIGURE 4-18. Intensity of Electroconvection Shadowgraph (III)

This figure shows the intensity pattern generated from a cell with the same parameters as in Figure 4-16 but with \( \theta_0 = .1 \). In this image the phase grating term dominates everywhere except on the surface of the cell (where it vanishes).


CHAPTER 5  Conclusions

5.1 How Well do the Theory and Experiment Agree?

Overall, the theory and experiment agree quite well. The absolute and convective instability boundaries show remarkable agreement. Moreover, the measured noise strength driving the convection rolls is consistent with the predictions of thermal noise to within a factor of five. Since the noise measurement is the result of extrapolation over several orders of magnitude in amplitude, this agreement is also quite good. However, the data analysis is quite involved and there are several potentially questionable steps. Firstly, the nonlinear theory is not directly applicable to the experiment. The amplitude equation has only been computed for an infinitely wide channel, whereas the experiment was obviously carried out in a finite box. It is clear that the finite width of the channel causes the convective instability boundary to be substantially altered from the infinite

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width case (See Section 3.7 on page 157). If the linear parameters such as
the convective instability boundary are so altered, it seems likely that the
nonlinear parameters would be similarly affected. Moreover the noise
strength is certainly altered by the presence of the lateral boundaries; an
issue which has not been adequately addressed in the literature. Another
problem is the effect of the curvature of the spiral on the convection rolls.
As important as are these effects, it is unlikely that they would change the
conclusion that the noise is thermal in origin. One would expect these
effects to change the noise strength by some amount of order unity. It
would be very unlikely that these geometrical factors would change the
noise strength by an order of magnitude. The hope is that, by using the
effective Reynolds number which is defined to have the same spatial
growth rate as the theory, these problems will be ameliorated. Of course
these issues would have to be resolved in order to make a more quantita-
tive comparison between the theory and the experiment.

5.2 How is it that the Thermal Fluctuations are Not Masked by
Laboratory Noise Sources?

One might imagine that external sources of noise in the laboratory would
dominate over the feeble effects of thermal noise. However, convection
with through flow is extremely insensitive to these noise sources. The cell
acts as a narrow band amplifier in the sense that only perturbations with
both the approximate critical spatial wavelength and the critical frequency will be amplified. The time scale for fluctuations which are amplified is set by the vertical diffusion time \( t_v = d^2/\kappa \) which for a 3 mm cell filled with water, is approximately 1 minute. Similarly, the length scale is set by the thickness of the cell. Typical laboratory sources of noise, including pumps and table vibrations, are acoustical. Acoustical noise with a wavelength near 3 mm is ultrasonic and therefore has the wrong time scale to couple into the convection. Similarly acoustical noise with the time scale of one minute has a length scale of many meters and again is strongly damped. More likely sources of noise are thermal in origin such as the temperature homogeneity of the boundary plates. The temperature on the plates can fluctuate due to imperfect mixing in the cooling water bath used to regulate the top plate temperature. However, since the plates have a large heat capacity, thermal variations in the bath, or voltage fluctuations on the bottom plate heater, with a one minute time scale will cause the plate temperatures to change almost uniformly along their length. The plates, in effect, act as low pass filters allowing only long (spatial) wavelengths and slow (temporal) fluctuations to penetrate to the sample fluid. Again, these fluctuations are strongly damped. The conclusion is that the typical laboratory sources of noise are all strongly damped and do not couple into the observed pattern. Only broad-band sources of noise
will have power in the band which is amplified. Brownian motion has a white noise power spectrum, and so has equal power in all bands.

The experimental sources of noise all tend to couple into the system as modulations of the order parameter $\epsilon$. This has been deemed *multiplicative* noise because in the CGL equation, $\epsilon$ is multiplied by the amplitude. The effect of fluctuations of $\epsilon$ depends on the state of the pattern. This is in contrast to the thermal noise which is an *additive* term to the amplitude equation. It has been shown theoretically that the effect of multiplicative noise is qualitatively different than additive noise in that it lowers the linear stability boundary whereas the additive noise does not [1]. Moreover, multiplicative noise couples in much more weakly than additive noise. In experiments in which the temperature difference across the cell has been modulated or ramped, Meyer \textit{et. al} found that the additive noise strength required to account for the measured heat current across the cell is of order $F = 5 \times 10^4 F_{\text{th}}$ [2]. The issue of why the additive noise in modulation experiments is so much larger than thermal noise, remains unresolved. However, Osenda \textit{et. al} found that if one attempts to account for the heat current measurements in terms of a multiplicative noise, one needs a noise power of at least $F = 10^9 F_{\text{th}}$, which is so large that it is clearly inconsistent with the measured temperature stability [3].
In the present experiment, the laboratory sources of additive noise are small, however, one could imagine nonlinear couplings of external noise which may become important. For example, consider the effect of a strong ultrasonic noise source which has the same wavelength as the critical mode. One could argue as before that the time scale of this drive is much too rapid to couple into the flow. However, a nonlinear effect such as the periodic temperature field generated by the dissipation of the waves, could have the same frequency as the critical mode and would therefore drive convection rolls. This suggests a simple experiment in which one would attach a piezoelectric oscillator to the bottom plate to generate the ultrasonic waves. Then one could look for rolls which are generated in response to the drive.

5.3 Future Directions

It is inevitable that performing one experiment suggests several others. The most obvious direction for future work is to repeat the experiment with different working fluids to verify if the noise scales as predicted by Eq 1-45. It is not very convenient to change the mean working temperature over a wide range, but it is certainly feasible to vary the density, the viscosity, the Prandtl number, and the thickness of the cell. Even in the absence of a more sophisticated theory, this would lend confidence to the result that rolls are driven by thermal noise. Another, simple experiment
which could be performed in the existing apparatus is to measure the statistical properties of the noise driven rolls. In the noise driven regime, the rolls grow to saturation downstream. Since they are saturated, their amplitudes vary little with time, but the phase of the rolls undergoes a random walk. This process is usually called “phase noise”. They dynamics of the phase noise in Taylor vortex flow has been studied by Babcock, Ahlers and Cannell [4]. It would be worthwhile to measure the dynamics of the phase noise in convection. Another fascinating area of study would be to measure the dynamics of the system near the co-dimension two point where transverse and longitudinal rolls exchange stability. It is clear that Rayleigh-Bénard convection, despite being heavily studied for decades, still has a lot of interesting new ground to cover.

5.4 Acknowledgments

Let me first express what a privilege it has been to work with my thesis advisors Guenter Ahlers, who taught me the value of choosing a problem of fundamental interest, and David Cannell, who more than anyone, taught me the fine art of instrument design. They truly have shown remarkable patience in letting me take the time to understand this experiment. I have also enjoyed working in the stimulating research group that they have assembled over the years. I have probably learned as much from the steady stream of students, postdocs, and visiting faculty as I have from
textbooks. Let me also thank the fine machinists in the UCSB physics machine shop for their skill in building parts of the apparatus and for teaching me how to build the rest. Finally let me thank my wife Gretchen. She has endured every leak, computer crash, and late night bicycle ride with soothing words of encouragement.

THE END


APPENDIX A  

Calculation of Baseflow for a Finite Width Channel

When the cell has a finite aspect ratio, the baseflow becomes substantially more complicated than for an infinite aspect ratio. In order for the solution of Eq 1-16 to satisfy the y boundary conditions in Eq 1-17, the velocity must be taken as a function of both y and z. Assume that Eq 1-16 is separable so that the most general form of the solution is

\[ v_t(x) = [Y(y)Z(z) + U(z) + V(y)] \hat{z}. \]

(A-1)

The solution for the pressure remains the same as Eq 1-28. The velocity equation becomes

\[ 0 = \frac{p_x}{Pr} + Y''(y)Z(z) + Y(y)Z''(z) + U''(y) + V''(z). \]

(A-2)

One can eliminate the \( p_x/Pr \) by letting \( U(y) = 0 \) and \( V(z) = -p_x(z^2 - \frac{1}{4})/Pr \) so that Eq 1-16 separates and becomes

\[ \frac{Y''(y)}{Y(y)} = - \frac{Z''(z)}{Z(z)}. \]

(A-3)

The left hand side of Eq A-3 is entirely a function of y, whereas the right hand side is a function of z. For this equation to be satisfied, both sides must be equal to a constant. Let this constant be \( c^2 \). The z portion of Eq A-3 becomes

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\[ Z''(z) = -c^2 Z(z) \]  \hspace{1cm} (A-4)

which has as solutions

\[ Z(z) = A \cos(cz) + B \sin(cz). \]  \hspace{1cm} (A-5)

In order to satisfy the boundary conditions that \( Z(\pm \frac{1}{2}) = 0 \), one gets the restrictions that \( B = 0 \) and \( c \) can take on only certain allowed values given by \( 0 = \cos\left(\frac{c}{2}\right) \) which requires

\[ c_n = (2n + 1)\pi. \]  \hspace{1cm} (A-6)

The \( y \) portion of Eq A-3 now becomes

\[ Y''(y) = c_n^2 Y(y), \]  \hspace{1cm} (A-7)

which has the solutions

\[ Y_n(y) = C_n \cosh(c_n y) + D_n \sinh(c_n y). \]  \hspace{1cm} (A-8)

however since the cell has a left right mirror symmetry, \( D_n = 0 \). Putting together the solutions Eq A-5 and Eq A-8 one gets

\[ v_l(x) = \left\{ \frac{D_z}{Pr} \left( \frac{1}{4} - z^2 \right) \right. \]
\[ + \sum_{n=0} A_n \cos((2n + 1)\pi x) \cosh((2n + 1)\pi y) \left\} \chi. \]  \hspace{1cm} (A-9)

One may determine the \( A_n \) from the horizontal boundary conditions \( v_l(y = \pm \frac{b}{2}) = 0 \). Inserting this into Eq A-9 gives

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\[
0 = \frac{p_x}{\Pr} \left( \frac{1}{4} - z^2 \right) + \sum_{n=0}^{\infty} A_n \cos[(2n+1)\pi z] \cosh \left[ (2n+1)\pi \frac{\beta}{2} \right]. \tag{A-10}
\]

One may use the orthogonality of the cosines to extract \( A_n \). Multiply Eq A-10 by \( \cos[(2m+1)\pi z] \) and integrate over \( z \). One finds

\[
A_m = \frac{p_x}{\Pr} \frac{8(-1)^{m+1}}{\pi^3 (2m+1)^3 \cosh \left[ (2m+1)\pi \frac{\beta}{2} \right]} \tag{A-11}
\]

Putting all of this together, one has the velocity as

\[
v_t (x) = \hat{x} \frac{p_x}{\Pr} \left\{ \left( \frac{1}{4} - z^2 \right) \right. \\
+ \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cos[(2n+1)\pi z] \cosh[(2n+1)\pi \frac{\beta}{2}]}{(2n+1)^3 \cosh[(2n+1)\pi \frac{\beta}{2}]} \left\} \right. \tag{A-12}
\]

One may now integrate this result over the channel cross section to compute the mean velocity

\[
\bar{v} = \frac{1}{\beta} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dy \int_{-1/2}^{1/2} dz |v(x)| \\
= \frac{p_x}{\Pr} \left\{ \frac{1}{6} - \frac{32}{\beta \pi^5} \sum_{n=0}^{\infty} \frac{\tanh[(2n+1)\pi \frac{\beta}{2}]}{(2n+1)^5} \right\}. \tag{A-13}
\]

Now one may use the mean velocity to re-express Eq A-13 in terms of the Reynolds number (recall that \( \bar{v} = \text{RePr} \))
\[ v_i(x) = \text{Re} \, \text{Pr} \, \mathcal{R} \{ \left( \frac{1}{4} - z^2 \right) + \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cos[(2n+1)\pi z] \cosh[(2n+1)\pi y]}{(2n+1)^3 \cosh[(2n+1)\pi \frac{\beta}{2}]} \} \times \left\{ \frac{1}{6} - \frac{32}{\beta \pi^5} \sum_{n=0}^{\infty} \frac{\tanh[(2n+1)\pi \frac{\beta}{2}]}{(2n+1)^5} \right\}^{-1} \] (A-14)

The maximum velocity is again attained at the center of the channel and is related to the mean velocity by

\[ v_{\text{max}} = \bar{v} \left\{ \frac{1}{4} + \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^3 \cosh[(2n+1)\pi \frac{\beta}{2}]} \right\} \times \left\{ \frac{1}{6} - \frac{32}{\beta \pi^5} \sum_{n=0}^{\infty} \frac{\tanh[(2n+1)\pi \frac{\beta}{2}]}{(2n+1)^5} \right\}^{-1} \] (A-15)

The following plots show the profile of the baseflow as the aspect ratio is changed. Although it is not obvious, this expression must have the symmetry \( v_{\text{max}}(\beta) = v_{\text{max}}(1/\beta) \) since the physical system has that symmetry. This is an interesting example of a mathematical theorem which is physically obvious, but mathematically obscure. The symmetry is easily demonstrated numerically. One finds that a plot of \( v_{\text{max}}(\beta) - v_{\text{max}}(1/\beta) \) simply yields 0 to within numerical round off errors. As expected, in the limit that the cell becomes very wide, \( v_{\text{max}}(\beta \to \infty) = \frac{3}{2} \bar{v} \), which recovers the previous result. A plot of the ratio \( v_{\text{max}}(\beta)/\bar{v} \) is shown in Figure A-1. For the cell used in the experiment one has

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FIGURE A-1. Ratio of Maximum Velocity to Mean Velocity in Channel as a Function of Aspect Ratio

This figure shows the ratio of the maximum velocity to the mean velocity as a function of the aspect ratio. This plot has the property that \( v_{\text{max}}(\beta) = v_{\text{max}}(1/\beta) \).

\( v_{\text{max}}(2.245) = 1.9577 \bar{v} \). This should be compared to \( v_{\text{max}}(\beta \to \infty) = \frac{3}{2} \bar{v} \). One finds that the maximum velocity in the cell differs from the infinitely wide case by only 31%. The velocity profile in the channel is shown in Figure A-2. The figure in the upper right corresponds to the aspect ratio of the experimental cell. It clearly has substantial variation in the y direction, unlike the large aspect ratio case.

Apparently 2.245 is not “close” to infinity.
FIGURE A-2. Velocity Profile of Base Flow

This figure shows the velocity distribution in the channel. The height of the plot is proportional to the velocity at that point in the channel. The y axis is across the page and the z axis is into the page. Each plot has been scaled so that maximum velocity is 1. Note that the y scales in each plot are different. The figure in the upper right corresponds to the aspect ratio of the experimental cell. When the aspect ratio becomes large, the flow is nearly parabolic in z, with a small healing length near the y boundaries. This is consistent with the infinite aspect ratio result.
APPENDIX B  

*Eigenfunctions of the Simple Bénard Problem*

With so many dimensionless parameters in the theory, it becomes difficult to compute the physical temperatures and velocities in the cell. This section will summarize the relationship between the amplitude in the complex Ginzburg-Landau equation and the physical fields for the simple Bénard problem (no applied flow). The expressions for the case with flow are analogous, if one replaces the baseflow and eigenfunctions with those computed by Luijx. I will again employ the convention that the physical fields will have a superscript "ph". All other quantities will either be (dimensioned) material parameters or (dimensionless) scaled fields. The temperature and velocity are given by the conduction state plus the deviation due to convection

\[ T^{ph}(x^{ph}, t^{ph}) = \frac{\Delta T^{ph}}{Ra} \left[ T_i(x^{ph}/d) + \theta(x^{ph}/d, \kappa x^{ph}/d^2) \right] \]
\[ v^{ph}(x^{ph}, t^{ph}) = \frac{\kappa}{d} \left[ v_i(x^{ph}/d) + v(x^{ph}/d, \kappa x^{ph}/d^2) \right], \]  

(B-1)

where \( T_i(x) \) and \( v_i(x) \) are the baseflow solutions and \( \theta(x, t) \) and \( v(x, t) \) are the deviations due to convection. For the simple Bénard problem

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\[
\frac{\Delta T_{ph}^{ph}}{Ra} T_{i}(x_{ph}/d) = \frac{T_{i}^{ph} + T_{b}^{ph}}{2} + \frac{T_{b}^{ph} - T_{i}^{ph}}{2} \left( \frac{z_{ph}}{d} \right) 
\] (B-2)

\[ v_{i}(x_{ph}/d, t) = 0. \]

The dimensionless fields $\theta$ and $v$ are given by

\[
\begin{bmatrix}
\theta(x, t) \\
v(x, t)
\end{bmatrix} = \left[ \frac{\bar{c} \bar{\theta}_0(z)}{d \bar{w}_0(z)} \right] \psi(x, y, t)
\] (B-3)

\[ \psi(x, y, t) = \frac{1}{\sqrt{2}} [A(x, y, t)e^{i q \cdot x} + c.c.], \]

and the amplitude normalization used is that given by Cross [1] (Eq 3)

\[ \langle \psi^2 \rangle = \frac{Ra}{Ra_c} (N - 1). \] (B-4)

The functions $\bar{\theta}_0$ and $\bar{w}_0$ are the lowest order eigenfunctions of the ODE's Eq 1-32 for $Re = 0$ which have been normalized so that their squares integrate to unity

\[
\begin{align*}
\bar{\theta}_0(z) &= \theta_0(z) \langle \theta_0(z)^2 \rangle^{-\frac{1}{2}} \\
\bar{w}_0(z) &= w_0(z) \langle w_0(z)^2 \rangle^{-\frac{1}{2}},
\end{align*}
\] (B-5)

where the angle brackets denote averaging across the layer. The functions $\theta_0(z)$ and $w_0(z)$ are the unnormalized eigenfunctions of Eq 1-32 computed by Schlüter, Lortz, and Busse [2]

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1. There is a slight numerical disagreement between Schlüter, et. al and Cross. In the former work, the argument of the cosine is given as 3.974 which is likely a typographical error. I have quoted the Cross value.

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\[ \theta_0(z) = \{ 650.68 \cos[3.9784 \ z] \\
+ (39.277 + 0.433 \ i) \cosh[(5.195 - 2.126i)z] \\
+ \text{c.c.} \} \]

\[ w_0(z) = 3.117^2 \{ \cos[3.9784 \ z] \\
+ (-3.076 + 5.195i) \times 10^{-2} \cosh[(5.195 - 2.126i)z] \\
+ \text{c.c.} \} . \]  \hspace{1cm} (B-6)

The normalization constants are

\[ \sqrt{\langle \theta_0(z)^2 \rangle} = 502.08 \]

\[ \sqrt{\langle w_0(z)^2 \rangle} = 5.8192. \]  \hspace{1cm} (B-7)

The constants \( \tilde{c} \) and \( \tilde{d} \) are given by integrals over the unnormalized eigenfunctions

\[ \tilde{c} = \left[ \frac{R_c \langle \theta_0(z)^2 \rangle}{\langle w_0(z) \theta_0(z) \rangle} \right]^{\frac{1}{2}} = 385.28 \]  \hspace{1cm} (B-8)

\[ \tilde{d} = \left[ \frac{R_c \langle w_0(z)^2 \rangle}{\langle w_0(z) \theta_0(z) \rangle} \right]^{\frac{1}{2}} = 4.4654. \]

Lastly, the magnitude of the critical wavevector has been computed by Schlüter, et. al. to have the value \( q_c = 3.117 \). For a coordinate system aligned with \( y \) axis along the rolls and the \( x \) axis parallel to \( k \) this reduces to
\[ T_{ph}^{\theta}(x_{ph}^{\theta}, z_{ph}^{\theta}) = \frac{T_b^{\theta} + T_i^{\theta}}{2} + \left( \frac{T_b^{\theta} - T_i^{\theta}}{2} \right) \left( \frac{z}{d} \right) + \frac{\Delta T_{ph}^{\theta}}{Ra} \sqrt{2} \zeta \tilde{\theta}_0 \left( \frac{z}{d} \right) \cos \left( q_c \frac{x}{d} \right) \]

\[ w_{ph}^{\theta}(x_{ph}^{\theta}, z_{ph}^{\theta}) = \frac{k}{d} \sqrt{2} \tilde{\theta}_0 \left( \frac{z}{d} \right) \cos \left( q_c \frac{x}{d} \right). \]

(B-9)

The Nusselt number and amplitude are related to the dimensionless \textit{rms} temperature and \textit{rms} \textit{z} velocity by

\[ \langle \theta^2 \rangle = \tilde{\epsilon}^2 \langle \psi^2 \rangle \]
\[ \langle w^2 \rangle = \tilde{d}^2 \langle \psi^2 \rangle. \]

(B-10)

where the angle brackets denotes averaging in the layer. Their dimensional counterparts are

\[ \theta_{rms}^{\theta} = \tilde{\epsilon} \frac{\Delta T_{ph}^{\theta}}{Ra} \sqrt{\langle \psi^2 \rangle} = \tilde{\epsilon} \Delta T_{ph}^{\theta} \sqrt{\frac{N - 1}{Ra Ra_c}} \]
\[ w_{rms}^{\theta} = \tilde{d} \frac{k}{d} \sqrt{\langle \psi^2 \rangle} = \tilde{d} \frac{k}{d} \sqrt{\frac{Ra}{Ra_c}} (N - 1). \]

(B-11)

A simple numerical example will help illustrate the magnitude of the temperatures and velocities generated by convection in this experiment. Evaluating Eq B-11 for the experiment, using the saturated amplitude of the theoretical value of the saturation amplitude \( A_{sat} = \sqrt{\epsilon/g} \), gives

\[ \theta_{rms}^{\theta} = 0.614^\circ C \sqrt{\epsilon} \]
\[ w_{rms}^{\theta} = 244 \frac{\mu m}{sec} \sqrt{\epsilon}. \]

(B-12)
where I have used the material parameters for water at the midpoint temperature in the cell \((T = 26.388 \degree C, \rho = 0.9962 \text{ gm/cm}^2,\alpha = 2.831 \times 10^{-4} /\degree C, \kappa = 1.481 \times 10^{-3} \text{ cm}^2/\text{sec}, \text{ and} \nu = 8.461 \times 10^{-3} \text{ cm}^2/\text{sec})\).

For a typical temperature difference which corresponds to \(\varepsilon = 0.1\) (\(\Delta T = 2.775 \degree C\) with a critical temperature difference of \(\Delta T_c = 2.523 \degree C\)), convection generates a \(\text{rms}\) temperature amplitude of 0.194\(\degree C\) and a \(\text{rms}\) velocity of 77 \(\mu\text{m/sec}\). Near the inlet, where the flow is noise driven, the saturated amplitude is of order \(A_{\text{sat}} \approx 10^{-4}\) so the \(\text{rms}\) temperature and velocity are \(\theta_{\text{rms}}^{\text{ph}} = 5 \times 10^{-5} \degree C\) and \(w_{\text{rms}}^{\text{ph}} = 20 \text{nm/sec}\).
APPENDIX C  Residuals of Fit to Retaining Pin Locations

The pin locations on the sapphire relative to the cell are shown in Figure C-1 and the measured residuals are listed in Table C-1.

FIGURE C-1. Cell Outline and Pin Locations
### Table C-1. Residuals of sapphire hole fit

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<th>Pin number</th>
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<td>2</td>
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<tr>
<td><strong>Average Error</strong></td>
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</table>
APPENDIX D

Calculation of Saturated Amplitude for Linearized GL Equation

In the sections of the cell where $\varepsilon$ is negative, one can compute a simple analytic form for the saturation amplitude of the linearized Ginzburg Landau equation. For the purposes of this appendix, I will neglect the small complex coefficients in the CGL and only consider the case where there is no applied flow and write:

$$\tau_0 \partial_t A(x, t) = \varepsilon A(x, t) + \xi_0^2 \partial_x^2 A(x, t) + f(x, t) \quad (D-1)$$

where $f(x, t)$ is a complex white noise term which is normalized so that

$$\langle f(x, t) \rangle = \langle f(x, t) f(x', t') \rangle = \langle f(x, t)^* f(x', t')^* \rangle = 0 \quad (D-2)$$

and the angle brackets represent integration over space and time. Since the amplitude is fluctuating, compute the mean square amplitude

$$\langle |A(x, t)|^2 \rangle.$$ Begin by rewriting Eq D-1 in Fourier space as

$$i \omega \tau_0 \tilde{A}(k, \omega) = (\varepsilon - k^2 \xi_0^2) \tilde{A}(k, \omega) + \tilde{f}(k, \omega). \quad (D-3)$$

This can be solved

$$\tilde{A}(k, \omega) = \frac{\tilde{f}(k, \omega)}{k^2 \xi_0^2 - \varepsilon + i \omega \tau_0}. \quad (D-4)$$
Multiply this by its complex conjugate to get

\[
\tilde{A}(k, \omega)\tilde{A}(k', \omega')^* = \frac{\tilde{f}(k, \omega)\tilde{f}(k', \omega')^*}{(k^2\xi_0^2 - \epsilon + i\omega\tau_0)(k'^2\xi_0^2 - \epsilon - i\omega'\tau_0)}.
\] (\(\text{D-5}\))

Integrate this over the \textit{primed} variables

\[
\langle |\tilde{A}(k, \omega)|^2 \rangle = 2\xi_0\tau_0 F \int \int \frac{dk d\omega}{(k^2\xi_0^2 - \epsilon)^2 + \omega^2\tau_0^2}.
\] (\(\text{D-6}\))

Now apply the Cauchy identity \(\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2} = \frac{\pi}{|a|}\) to perform the \(\omega\) integral

\[
\langle |\tilde{A}(k, \omega)|^2 \rangle = \frac{2\pi^2 F}{\sqrt{|\epsilon|}}
\] (\(\text{D-7}\))

Finally, use Parseval's theorem to convert the Fourier amplitude to the real space amplitude

\[
\langle |A(x, t)|^2 \rangle = \frac{1}{(2\pi)^2} \langle |\tilde{A}(k, \omega)|^2 \rangle = \frac{F}{2\sqrt{|\epsilon|}}.
\] (\(\text{D-8}\))
E.1 Introduction

The program thermit, performs thermal modeling in a rectangular 2D geometry. One can think of it as modeling a slab of material that is finite in x and y and infinite in z. The program supports arbitrary numbers of blocks of materials with differing thermal conductivities. It allows insulating regions, constant temperature regions, and constant power regions. One can specify the regions using a simple language that has several commands. This makes it ideal for quickly analyzing the thermal properties of proposed pieces of apparatus. The heart of the program is a fixed grid relaxation algorithm [3].

E.2 The program commands

One invokes the program with the command:

thermit [-(n)pts=10] [-(e)psilon=.01] [-(v)erbose] [-(r)eport=20]

The brackets denote optional parameters and the default values are shown. This command defines \( n_{pts} \) points in the shortest direction and a convergence criterion of epsilon. It also turns on the verbose flag which
causes the program to print epsilon every report iterations of the loop. The program will iterate until the rms change of the cells is less than epsilon. When one has slow transients this will not be sufficient so one may want to decrease the convergence criterion. The program uses a relaxation algorithm and so the run time is \( O(\log e) \). This means that squaring the convergence criterion to 0.0001 only doubles the run time. I am not sure of the scaling with \( npts \) but I think it is \( O(npts^2) \). After thermit is invoked from the command line it enters a loop waiting for commands to define the geometry. One can type the commands from the keyboard but typically one makes an initialization file and invokes the program with:

\[
\text{nice 20 thermit < initialization_file > output_file &}
\]

This has the effect of running the program in the background at low priority. It takes the commands from initialization_file and writes the output to output_file.

The commands available are:

- \text{scale} \quad xscale\, yscale
- \text{insul} \quad \text{xbegin }\, ybegin\, \text{xend }\, \text{yend}
- \text{region} \quad \text{xbegin }\, ybegin\, \text{xend }\, \text{yend }\, \text{lambda}
- \text{temp} \quad \text{xbegin }\, ybegin\, \text{xend }\, \text{yend }\, \text{temp}
- \text{power} \quad \text{xbegin }\, ybegin\, \text{xend }\, \text{yend }\, \text{power}
- \text{init} \quad \text{temp}
dump
dumpall
end

insulating is a synonym for insul.

The first thing one must do is set the unit scale of the simulation. The command is:

scale   xscale yscale

Where xscale and yscale are double precision numbers and default to 1.0. Once this command is issued all subsequent commands use these units. The coordinate system will then range from (0, xscale) and (0, yscale). The grid is sized so that there are npts in the shortest direction.

One must next define the conducting regions with:

region   xbegin ybegin xend yend lambda

One may define as many overlapping regions as one wishes. Of course, regions defined later overlay the earlier regions. The default region (if one is not defined) is conductivity 1 and covers all space. Similarly one can define conducting regions or boundaries with:

insul   xbegin ybegin xend yend

It is permitted to have a degenerate region which has 0 width. This is useful for defining boundary conditions. For example: insulating 0 0 1 0 defines a insulating lower boundary. If an insulating region has a non-zero
internal area, these points are not involved in the thermal iteration (they can't change) and are frozen at their initial values of $T=0$. The edges of the insulating regions are set to the local temperature of the neighboring points.

Constant temperature regions are defined with

```
temp     xbegin ybegin xend yend temp
```

with similar treatment to the insulating case. Here the temperature is set to `temp`. These points are obviously not changed in the iterations. The last case is constant power.

```
power    xbegin ybegin xend yend temp
```

which sets the local derivative of `temp`. Currently ONLY degenerate regions are supported. All of the power is dissipated on the boundaries of the region. There is one caveat in using constant power boundary conditions. They converge much more slowly than constant temperature boundary conditions.

The command

```
init      temp
```

is optional. It sets all non constant grid cells to `temp`. Otherwise they initialize to 1. If `temp` is suitably chosen, this can have the effect of speeding the convergence of the relaxation. This is particularly important when
injecting constant power into a nearly perfect conductor. If temp is chosen
to be near the final temp of the constant power boundary, it will have a
dramatic effect. Usually it is good to run a fast low precision (i.e. few
points) run to determine an appropriate value of temp.

Then there are a couple of diagnostic routines: dump, dumpall, end.
Dump dumps the array of boundary conditions. Each location has a
number which specifies the type of iteration used at the node. There are 6
cases listed in Table E-1. Types of Nodes on page 270

<table>
<thead>
<tr>
<th>Node #</th>
<th>Type of Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>regular point</td>
</tr>
<tr>
<td>1</td>
<td>general point</td>
</tr>
<tr>
<td>2</td>
<td>horizontal boundary</td>
</tr>
<tr>
<td>3</td>
<td>vertical boundary</td>
</tr>
<tr>
<td>4</td>
<td>constant temperature</td>
</tr>
<tr>
<td>5</td>
<td>constant power</td>
</tr>
</tbody>
</table>

This is really only useful for debugging the model. It often happens that
one defines the regions in the wrong order and a corner points of the
overlapping region.

An example is helpful. Anything after a '"#' is considered a comment.

```
$ thermit -n 10 -eps.0001
scale 1 1  # define unit scale
region 0 0 1.5  # Error (No conductivity)
```
region called with wrong number of arguments (4)
region 0 0 1.5 1  # Define the lower half plane to be conductivity 1
region 0.5 1 1 5  # Define the upper half plane to be conductivity 5
insulating 0 1 1 1  # Make the right wall insulating
temp 0 0 0 1 0   # make the bottom T=0
temp 0 0 1 0 0   # Make the left wall T=0
temp 0 1 1 1 1    # Make the top wall T=1
dump             # Examine the array of boundary conditions. (See below)
444444444444
444444444444
440000000030
440000000030
440000000030
442222222210
440000000030
440000000030
440000000030
440000000030
440000000030
444444444444
444444444444
end             # Start the simulation. The bottom is the result
                # Relax, drink some coffee.
10              # Returned length of lines (depends on aspect ratio)
0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
0.0000 0.0291 0.0556 0.0773 0.0926 0.1007 0.1007 0.0916 0.0720 0.0410
0.0000 0.0609 0.1160 0.1608 0.1925 0.2095 0.2105 0.1936 0.1556 0.0918
0.0000 0.0985 0.1867 0.2574 0.3071 0.3343 0.3381 0.3167 0.2651 0.1708
0.0000 0.1465 0.2749 0.3751 0.4442 0.4825 0.4910 0.4699 0.4172 0.3262
0.0000 0.1670 0.3095 0.4174 0.4901 0.5300 0.5394 0.5195 0.4707 0.3957
0.0000 0.2120 0.3785 0.4949 0.5687 0.6079 0.6171 0.5981 0.5502 0.4721
0.0000 0.3023 0.4978 0.6149 0.6821 0.7157 0.7230 0.7057 0.6599 0.5756
0.0000 0.4995 0.6956 0.7849 0.8292 0.8496 0.8536 0.8419 0.8081 0.7306
1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000

When more points are added (npts=50) Mathematica produced the contour plot shown in Figure E-1.

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a = ReadList["output_file", Number];
dim = a[[1]]; 
a = Partition[Drop[a, 1], dim];
ListContourPlot[a];

FIGURE E-1. Contour Plot of Temperature Field
This figure shows a contour plot the temperature field for the example described in the text.

E.3 A Word On The Order Of Commands
It may seem that the order of defining non-overlapping regions is arbitrary, but this isn't quite true. The problem comes from the what to do with the elements that border two regions. For conducting regions the borders are set to the average conductivity of the two regions. For the
insulating region the conductivity is set to 0. If one defines an insulating region before defining an abutting conducting regions, there will be a row of elements with half of the conductivity (the average of 0 and lambda). This is not usually what one wants. To avoid these sorts of problems, I usually define the conducting regions first, then the insulating regions, then the constant power and constant temp regions. This avoids the problem. Of course it is always a good idea to dump the boundary condition array for a small matrix and check that it is really doing what you think it is doing.

E.4 Controlling The Running Program

Once the program is off running in the background there are still a couple of things the user can do to affect its operation. One can always kill the process with kill. The unix kill command, however, is more flexible that. It can send arbitrary unix signals. The program catches signal SIGUSR1 and interprets this as a command to print the current epsilon of convergence on the console. One can invoke this with:

kill -USR1 pid

where pid is the process id of the background job.
The program also catches SIGUSR2 and interprets this to mean, "dump the current temperature array into a file called foo.log." As before one can invoke this with:

```
kill -USR2 pid
```
APPENDIX F

The Effect of Windowing on DFT Power Spectra

In the course of the data analysis extensive use is made of power spectra of time series and their moments. These are computed by taking a sampled time series, multiplying it by a windowing function, and then computing its discrete Fourier transform (DFT). The square of the DFT is the power spectrum. In this appendix the effect of the windowing function will be investigated. In particular, it is well known that when the length of the time series is not an integral multiple of the period of the underlying physical process being sampled, the power spectrum is artificially broadened. This is often called spectral leakage and it directly affects the moments of the spectral peaks. If one is to make sense of measurements of the moments of spectral peaks, the effect of the windowing function must be understood. In this work, I use the Hanning windowing function

$$w(x) = \frac{1}{2} \left[ 1 - \cos \left( 2\pi \frac{x}{P} \right) \right].$$  \hspace{1cm}(F-1)

where $P$ is total length of the time series. Consider what happens when one samples a periodic signal. The DFT assumes that the data set is periodic with a period equal to the length of the data set. If the sample does not contain an integral number of cycles, the end of the sample is not aligned with the beginning and one gets a discontinuity at the end of the
data set. This is illustrated schematically in Figure F-1. The discontinuity introduces high frequency ringing into the DFT power spectrum. The purpose of the windowing function is to insure that the sampled data set goes to nearly zero at the ends in an effort to reduce the magnitude of the discontinuity. It also has the effect of smearing out peaks in the power spectrum. This is easily understood. Let \( f_n \) be the sampled time series and \( F_k \) its DFT. We can write the windowed data set as \( g_n = w_n f_n \) where \( w_n \) is the sampled window function. The DFT of \( g_n \) is just the
convolution of the DFT of the window function and the DFT of the sample.

\[ G_k = W_k \otimes F_k \]  \hspace{1cm} (F-2)

If one uses the Hanning window function then one may write its DFT explicitly as

\[ W_k = \frac{1}{2} \delta_{k,0} - \frac{1}{4} \delta_{k,-1} - \frac{1}{4} \delta_{k,1} \]  \hspace{1cm} (F-3)

so that one may write

\[ G_k = \frac{1}{2} F_k - \frac{1}{4} F_{k-1} - \frac{1}{4} F_{k+1} \]  \hspace{1cm} (F-4)

which shows that the effect of the windowing function is to smear out the peaks in Fourier space.

Consider the change in the total power introduced by windowing. Parseval's theorem states that the \textit{rms} power in real space is equal to the \textit{rms} power in Fourier space. How does this relate to the \textit{rms} power of the windowed function. The discrete form of Parseval's theorem is

\[ \sum_{k=0}^{N-1} |F_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |f_n|^2 . \]  \hspace{1cm} (F-5)

The sum of the power of the windowed function is now given by
\[
\sum_{k=0}^{N-1} |G_k|^2 = \sum_{k=0}^{N-1} \left| \frac{1}{2} F_k - \frac{1}{4} F_{k-1} - \frac{1}{4} F_{k+1} \right|^2 \\
= \sum_{k=0}^{N-1} \frac{1}{4} |F_k|^2 + \frac{1}{16} |F_{k-1}|^2 + \frac{1}{16} |F_{k+1}|^2 \\
+ \frac{1}{8} (F_k F_{k-1}^* + F_k F_{k+1}^* + F_{k-1} F_{k+1}^* + \text{c.c.}).
\]

Re-index the sums to combine terms. One finds

\[
\sum_{k=0}^{N-1} |G_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} \frac{3}{8} |f_n|^2 + \sum_{n=0}^{N-1} \left( \frac{1}{4} F_k F_{k-1}^* + \frac{1}{8} F_{k-1} F_{k+1}^* + \text{c.c.} \right).
\]

Apply this to the problem of sampling a unit sine wave which is not necessarily commensurate with the sample interval. For convenience, let the sample interval be 1. One then has

\[
f(t) = \sin(2\pi f t).
\]

where \( f \) is not necessarily integral. If \( f \) is integral, then the last sum (in parentheses) vanishes and one finds that the total power is reduced by 3/8. The last sum, however, does not vanish in general and Parseval's theorem is violated by the windowing in the sense that the \textit{rms} power in the time domain is no longer simply related to the frequency domain.

Next turn to the extent of the ringing introduced to the power spectrum when \( f \) is non-integral. To simplify the calculations I will consider the data set to be continuously sampled and compute the Fourier series instead of the DFT.
\[ f(t) = b_0 + \sum_{n=1}^{\infty} a_n \sin \left( n\pi \frac{t}{P} \right) + b_n \cos \left( n\pi \frac{t}{P} \right). \]  \hfill (F-9)

where \( a_n \) and \( b_n \) are the Fourier coefficients. They are given by the orthogonality relations of the sine and cosine and can be computed analytically for our sample problem for both the windowed and un-windowed time series

\[
\begin{align*}
  a_n &= \frac{2}{P} \int_{0}^{P} \sin \left( n\pi \frac{x}{P} \right) f(t) dt \\
  b_n &= \frac{2}{P} \int_{0}^{P} \cos \left( n\pi \frac{x}{P} \right) f(t) dt \\
  b_0 &= \frac{1}{P} \int_{0}^{P} f(t) dt.
\end{align*}
\]  \hfill (F-10)

One can then evaluate the Fourier coefficients for the un-windowed sample

\[
\begin{align*}
  a_n &= \frac{n \sin(2\pi f)}{\pi f^2 - n^2} \\
  b_n &= \frac{2 f \sin^2(2\pi f)}{\pi f^2 - n^2} \\
  b_0 &= \frac{\sin^2(\pi f)}{\pi f}.
\end{align*}
\]  \hfill (F-11)

Of course, when \( f \) is integral, one has the trivial result \( a_n = \delta_{n,f} \) and \( b_0 = b_n = 0 \). The resulting power spectrum is given by the square of the Fourier coefficients \( P_n = a_n^2 + b_n^2 \). When one applies the Hanning window one finds the coefficients

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\[ a_n = \frac{n}{2\pi (1-n-f)(1+n-f)(n-f)(1-n+f)(n+f)(1+n+f)} \left( \frac{n^2 + 3f^2 - 1}{\sin(2\pi w)} \right) \]

\[ b_n = \frac{f}{\sin^2(2\pi f)(1-n-f)(1+n-f)(n-f)(1-n+f)(n+f)(1+n+f)} \left( \frac{n^2 + 3f^2 - 1}{\sin(2\pi w)} \right) \]

\[ b_0 = \frac{\sin^2(2\pi f)}{2\pi f(1-f^2)}. \]

One may now compute the moments of the power spectrum as a function of the applied frequency \( f \). The first moment of the peak can be considered an experimental estimate of the true frequency.

\[ \langle n \rangle = \frac{\sum_{n=0}^{\infty} n P_n}{\sum_{n=0}^{\infty} P_n}. \]  \hspace{1cm} (F-13)

It is straightforward to evaluate this for the example of the unit sine wave coefficients given in Eq F-11 but the resulting expressions are complicated and will not be presented. When the \( f \) is integral, the first moment is exactly \( f \) since

\[ \langle n \rangle = \sum_{n=0}^{\infty} n \delta_{n,f} = f. \]  \hspace{1cm} (F-14)

Similarly the first moment for the windowed time series is

\[ \langle n \rangle = \frac{\sum_{n=0}^{N-1} n \left| \frac{1}{4} \delta_{n,f} - \frac{1}{4} \delta_{n,f-1} - \delta_{n,f+1} \right|^2}{3/8} \]

\[ = \frac{8}{3} \left[ \frac{f}{4} + \frac{f-1}{16} + \frac{f+1}{16} \right] \]

\[ = f. \]  \hspace{1cm} (F-15)
When \( f \) is not integral there is a surprise. Consider the asymptotic behavior of the power spectrum for large values of \( n \). The un-windowed Fourier coefficient \( a_n \sim 1/n \) so the power spectrum \( P_n \sim 1/n^2 \). Therefore the computation of the first moment in Eq F-13 involves the harmonic sum \( \sum_{n=0}^{\infty} 1/n \) which diverges! The first moment isn't an estimate of the underlying frequency, it is infinite. The corresponding calculation for the windowed Fourier coefficients are much better behaved. The asymptotic power spectrum goes as \( P_n \sim n^{-6} \) so the sums in Eq F-13 converge.

The results are evaluated numerically in Figure F-2 which shows the first moment as a function of the underlying frequency \( f \). If the first moment was a good estimate, this would be a straight line with unit slope. The windowed method rapidly approaches this asymptote, but the un-windowed moment rings wildly between the integral frequencies (where it is exact).

One may attribute the divergence of the first moment of the un-windowed power spectrum to the fact that the Fourier series has an infinite number of terms. When one has a sampled data set, the DFT is truncated to a finite number of terms and all power at the higher frequencies is aliased back to lower frequencies, effectively removing the divergence. However, the problem does not disappear. Consider what happens if one takes a finite sample and computes the first moment. One gets some value
FIGURE F-2. First Moment of a Unit Sine Wave vs. Applied Frequency

The first moment is plotted against the frequency of a unit sine wave. If the first moment is to be interpreted as a measure of the frequency, then this graph should be a straight line with unit slope. The two dashed lines are the results for an un-windowed sample with 100 and 1000 terms respectively. The solid line is the calculation for the windowed data for 100 terms. Both methods give an exact result when \( f \) is integral and larger than 1. However, the windowed method rapidly approaches a straight line, but the un-windowed method oscillates. What is more disturbing is that the amplitude of the oscillation grows as the number of terms in the sum is increased. See Figure F-3.

which is near \( f \). Then increase the number of samples by sampling more frequently, but keep the total sample interval constant. This increases the Nyquist frequency so more terms are retained in the Fourier series. Less power is aliased to lower frequencies and a correspondingly higher value for the moment is obtained. One is led to the paradoxical result that the
more rapidly one samples, the less accurately the first moment measures the frequency of the underlying time series. This effect is illustrated in Figure F-3. A pure sine wave with \( f = 10.75 \) was sampled with increasing number of samples. An FFT of the resulting time series was performed and the first moment of the power spectrum was computed. Again, the windowed moment rapidly approaches the applied frequency. The un-
windowed moment shows the peculiar divergence that causes more sampling to generate a poorer estimate of the frequency.

Measurements of the second moment are similarly affected by windowing. The second moment should be zero since the sample function is a pure sinusoid. However due to the finite sampling interval and the associated discontinuity, it has a finite width. It is given by

\[
\sigma^2 = \langle n^2 \rangle - \langle n \rangle^2 = \frac{\sum_{n=0}^{\infty} n^2 P_n}{\sum_{n=0}^{\infty} P_n} - \langle n \rangle^2.
\] (F-16)

The second moment also diverges for the un-windowed system, however it diverges more rapidly than the first moment since the sum involves \( n^2 P_n \sim O(1) \). Similarly the second moment converges for the windowed system. As before, we can evaluate these expressions when \( f \) is integral. The un-windowed second moment is

\[
\sigma^2 = \frac{\sum_{n=0}^{\infty} n^2 \delta_{n,f}}{\sum_{n=0}^{\infty} \delta_{n,f}} - \langle n \rangle^2 = f^2 - f^2 = 0.
\] (F-17)

The windowed second moment is

\[
\sigma^2 = \frac{\sum_{n=0}^{\infty} n^2 \left( \frac{1}{4} \delta_{n,f} + \frac{1}{16} \delta_{n,f-1} + \frac{1}{16} \delta_{n,f+1} \right)}{\sum_{n=0}^{\infty} \delta_{n,f}} - \langle n \rangle^2
= \frac{8}{3} \left( \frac{1}{4} f^2 + \frac{1}{16} (f-1)^2 + \frac{1}{16} (f+1)^2 \right) - f^2
= \frac{1}{3}.
\] (F-18)
Note that the windowed second moment is independent of frequency.

Numerical results for non-integral frequencies are presented in Figure F-4. It appears that for the windowed time series, the second moment is

![Graph showing second moment as a function of frequency](image)

**FIGURE F-4. Numerical Evaluation of Second Moment as a Function of Frequency**

This figure shows the second moment as a function of frequency for both the windowed and un-windowed time series. The Fourier series were evaluated with 100 terms. The un-windowed second moment diverges and so is a poor estimate of the spectral frequency spread.

actually independent of $f$ for all frequencies above $f \geq 1.5$ although this is not readily apparent from the analytic sums in Eq F-16. Since

$\sigma = \sqrt{1/3}$ , for a pure sine wave, when investigating the spectral spread of some unknown function which is nearly sinusoidal, I actually compute
\( \sigma^{\text{eff}} = \sigma - \sqrt{1/3} \) in the analysis of the absolute stability boundary. See *Absolute Stability Boundary* on page 178.
APPENDIX G  \textit{Inversion of the Ray Path}

In the computation of the Eikonal for rays passing through a convection cell, one computes the Eikonal on the top surface of the cell as a function of the $x$ position of the entering ray $x_0$. However, to match boundary conditions on the top surface, one needs this function to be parametrized by the $x$ position of the exiting ray $x_1$. The ray equation gives us the relation between $x_1$ and $x_0$ as a power series in the small parameter $n_1$.

Write this relation as

$$x_1 = x_0 + f(x_0)n_1 + g(x_0)n_1^2 + O(n_1^3). \quad (G-1)$$

Hence we are faced with the problem of inverting this relation when the functional forms $f(x)$ and $g(x)$ are known. Write the inverse relation as a power series in $n_1$,

$$x_0 = x_1 + a(x_1)n_1 + b(x_1)n_1^2 + O(n_1^3). \quad (G-2)$$

where the functions $a(x)$ and $b(x)$ are to be determined in terms of $f(x)$ and $g(x)$. Insert Eq G-2 into Eq G-1 to get

$$x_1 = x_1 + a(x_1)n_1 + b(x_1)n_1^2 + O(n_1^3) + f(x_1 + a(x_1)n_1 + b(x_1)n_1^2 + O(n_1^3))n_1 + g(x_1 + a(x_1)n_1 + b(x_1)n_1^2 + O(n_1^3))n_1^2. \quad (G-3)$$

Expand out $f(x)$ and $g(x)$ in a Taylor series about $x_1$ to get
\begin{align}
0 &= [a(x_1) + f(x_1)]n_1 \\
   &+ [b(x_1) + f'(x_1)a(x_1) + g(x_1)]n_1^2 + O(n_1^3). \tag{G-4}
\end{align}

The coefficient of each power of $n_1$ must vanish. Solve for $a(x)$ and $b(x)$ and substitute back into Eq G-2 to get

\begin{align}
x_0 &= x_1 - f(x_1)n_1 + [f(x_1)f'(x_1) - g(x_1)]n_1^2 + O(n_1^3). \tag{G-5}
\end{align}
APPENDIX H

Fourier Transform of the Eikonal Solution

Assume one has computed the electric field on the boundary of the cell and has found it to have the form

\[ E(x) = E_0 (a_0 + a_1 \cos(Kx) + a_2 \cos(2Kx)) \times \]
\[ e^{iK(\phi_0 + \phi_1 \cos(Kx) + \phi_2 \cos(2Kx))} e^{-iKx} \]  \hspace{1cm} (H-1)

In this expression \( a_1 \) and \( a_2 \) represent the magnitudes of the amplitude gratings and \( \phi_1 \) and \( \phi_2 \) are the phase gratings. One can compute the Fourier transform of this solution by using the Bessel function identity

\[ e^{iK \cos(Kx)} = \sum_{l=-\infty}^{\infty} J_l(\tilde{k} \phi) e^{i[l(Kx + \frac{\pi}{2})]} \]  \hspace{1cm} (H-2)

to rewrite the exponentials in Eq H-1. Also rewrite the cosine terms as sums of exponentials. One finds

\[ E(x) = E_0 e^{i(K_0 - \omega x)} \left[ a_0 + \frac{a_1}{2} (e^{iKx} + e^{-iKx}) + \frac{a_2}{2} (e^{2iKx} + e^{-2iKx}) \right] \times \]
\[ \left[ \sum_{l=-\infty}^{\infty} J_l(\tilde{k} \phi_1) e^{i[l(Kx + \frac{\pi}{2})]} \right] \left[ \sum_{m=-\infty}^{\infty} J_m(\tilde{k} \phi_2) e^{i[m(2Kx + \frac{\pi}{2})]} \right] \]  \hspace{1cm} (H-3)

Multiply out the products and collect the powers of \( e^{iKx} \). Define \( n \equiv l + 2m \) to get
\[ E(x) = E_0 e^{i(\bar{\kappa}\phi_0 - \omega x)} \sum_{n=-\infty}^{\infty} E_n e^{in\bar{\kappa}x} \]

\[ E_n = \sum_{m=-\infty}^{\infty} J_m(\bar{k}\phi_2) e^{i(n-m)\frac{\pi}{3}} \left[ a_0 J_{n-2m}(\bar{k}\phi_1) ight. \]
\[ \left. + i \frac{a_1}{2} (J_{n+1-2m}(\bar{k}\phi_1) - J_{n-1-2m}(\bar{k}\phi_1)) \right. \]
\[ \left. - \frac{a_2}{2} (J_{n+2-2m}(\bar{k}\phi_1) + J_{n-2-2m}(\bar{k}\phi_1)) \right]. \]

This form can now be easily matched to the free solutions in Eq 4-40.
APPENDIX I

Effective Index of Refraction for Uniaxial Crystals

Consider a ray of light travelling through a uniaxial crystal at an angle to the optical axis as shown in Figure I-1. Choose the local coordinate system with the optical axis along \( x \), \( z \) vertical, and \( y \) into the plane. The angle between \( \mathbf{k} \) and \( \mathbf{\hat{x}} \) is \( \gamma \). Since the material is anisotropic, \( E \) and \( D \) are, in general, not parallel. First consider the polarization with \( E \) in the \( x-y \) plane. Let \( \alpha \) and \( \beta \) be the angles from \( \mathbf{\hat{x}} \) to \( E \) and \( D \) respectively. We will look for plane wave solutions to the wave equation of the form

\[
E(x, t) = E e^{i(k \cdot x - \omega t)} \quad \text{and} \quad D(x, t) = D e^{i(k \cdot x - \omega t)}.
\]

Since \( \nabla \cdot D = 0 \), we
have $\mathbf{k} \cdot \mathbf{D} = 0$ which implies $\beta + \gamma = \pi/2$. In this coordinate system the dielectric tensor is

$$
\varepsilon = \begin{pmatrix}
\varepsilon_e & 0 & 0 \\
0 & \varepsilon_o & 0 \\
0 & 0 & \varepsilon_o
\end{pmatrix},
$$

so that one may write

$$
\tan \beta = \frac{D_y}{D_x} = \frac{\varepsilon_o E_y}{\varepsilon_e E_x} = \frac{\varepsilon_o}{\varepsilon_e} \tan \alpha.
$$

The wave equation was given in Eq 4-26 as

$$
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\varepsilon \mathbf{E}) = \nabla^2 \mathbf{E} - \nabla (\nabla \cdot \mathbf{E}).
$$

which implies

$$
-\frac{\omega^2 c^2}{\varepsilon} \varepsilon \mathbf{E} = -k^2 \mathbf{E} + \mathbf{k} (\mathbf{k} \cdot \mathbf{E}).
$$

Square both sides to get the scalar equation

$$
\frac{\omega^4}{c^4} \left( \varepsilon_o^2 \sin^2 \alpha + \varepsilon_e^2 \cos^2 \alpha \right) = k^4 \sin^2 (\alpha + \beta).
$$

This can be solved to get the effective index of refraction

$$
n_{\text{eff}} \equiv \frac{k c}{\omega} = \left[ \frac{\varepsilon_o \varepsilon_e}{\varepsilon_o \sin^2 \gamma + \varepsilon_e \cos^2 \gamma} \right]^{1/2}.
$$

Return to the case of the polarization with $\mathbf{E}$ polarized along the $y$ axis. Eq I-4 implies $\omega^4 \varepsilon_o^2 / c^2 = k^4$ so that $n_{\text{eff}} = n_o$. 

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