Effects of Finite Geometry on the Wave Number of Taylor-Vortex Flow

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We report measurements of the axial variation of the wave number \( q \) of Taylor-vortex flow in a system with aspect ratio \( 17 \leq L \leq 25 \) containing ten vortex pairs between rigid nonrotating ends. Near the critical Reynolds number \( R_c \), \( q \) is very nonuniform when its average value \( \bar{q} \) differs significantly from its critical value \( q_c \). For sufficiently small \( |\bar{q} - q_c| \), the finite geometry eliminates the Eckhaus instability. Our results agree quantitatively with solutions of the Ginzburg-Landau equation.

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Pattern formation is a central problem in the study of nonlinear systems under stress. Extensively studied examples, both experimentally and theoretically, are Rayleigh-Bénard convection and Taylor-vortex flow (TVF). Of these, the latter is particularly amenable to quantitative study, because the velocity field varies independently only in the \( z \) direction along the cylinder axis. The pattern-formation problem is thus reduced to the determination of the axial variation of the wave number \( q(z) \). For an infinite TVF system, \( q \) would be uniform. Its value would be somewhere in the range corresponding to states of the system which are stable against long-wavelength axisymmetric perturbations (the Eckhaus stable band).4 However, real systems have physical boundaries at \( z = 0 \) and \( z = L \), and usually5 contain an integer number \( N \) of vortex pairs. The influence of the boundaries and of the integer vortex number on the pattern \( q(z) \) is of great interest.

We studied a rotating Couette-Taylor system with rigid, nonrotating boundaries. In that case, the bifurcation from axially uniform Couette flow to TVF is imperfect.3 For all Reynolds numbers \( R > 0 \), the boundaries drive vortices adjacent to themselves. For \( R \) less than the critical value \( R_c \) for the onset of TVF in the infinite system, the amplitudes of these vortices decay exponentially with distance from the boundaries; but for \( R \geq R_c \) the entire system is filled with vortices of significant strength. It is well known that the vortex nearest a boundary (the Ekman vortex) has an anomalous width and strength.3 We find that the amplitude \( |A(z)| \) and the wave number \( q(z) \) of the vortices in the system \textit{interior} are very uniform when the average wave vector \( \bar{q} \) (excluding the Ekman vortices) is close to the value \( q_c \) at the onset of TVF for \( L = \infty \). However, when \( L \) and \( N \) are such that \( \bar{q} \) differs substantially from \( q_c \), then both \( |A(z)| \) and \( q(z) \) are significantly dependent upon \( z \). The observations agree with predictions based on the Ginzburg-Landau amplitude equation,6-8 which include1-3 that \( |A(z)|^2[q(z) - q_c] \) is \( z \) independent. The present work differs from an earlier study by Pfister and Rehberg9 in that all their measurements were carried out at \( \bar{q} = q_c \).

When \( \bar{q} \) differs substantially from \( q_c \), a quasistatic decrease of \( R \) causes a transition when the Eckhaus boundary is crossed.4,10,11 For \( \bar{q} > q_c \) a vortex pair is lost, while for \( \bar{q} < q_c \) one is gained. In the finite system we find that the Eckhaus instability ceases to exist when \( |\bar{q} - q_c| \) is sufficiently small. The gap in the Eckhaus boundary is just large enough to avoid a transition from an unstable state at \( \bar{q} < q_c \) (\( \bar{q} > q_c \)) to another unstable state at \( \bar{q} > q_c \) (\( \bar{q} < q_c \)) due to the gain (loss) of a vortex pair. A gap of about this width in the Eckhaus instability of the finite periodic system was predicted for the Ginzburg-Landau equation10-8 by Kramer and Zimmermann.11

We studied states consisting of ten vortex pairs between rigid nonrotating axial boundaries. One boundary was movable to allow the system length \( H \) to be adjusted. The fluid was 43% by volume glycerol in water with \( 3 \) \( \mu \text{g/cm}^3 \) of \( 1.09-\mu\text{m-diam polystyrene latex spheres} \) to allow laser Doppler velocimetry. It was confined between two concentric horizontal cylinders with the inner one, of radius \( r_1 = 1.867 \) cm, rotating, and the outer one, of inner radius \( r_2 = 2.499 \) cm, stationary. The aspect ratio \( L = H/(r_2 - r_1) \) was varied between 17 and 25, with an accuracy of \( \pm 0.2\% \). The temperature was controlled to \( \pm 5 \) mK at \( 21^\circ \text{C} \). We measured the axial component of the fluid velocity \( v_z \) at a radial position about 1 mm from the inner cylinder as a function of axial position \( z \) (measured in units of \( r_2 - r_1 \)) and of \( \epsilon = (R - R_c)/R_c \). The flows were created either through a quasistatic increase in \( R \) at fixed \( L \) or by adjustment of \( L \) once such a flow was obtained. All measurements pertain to steady states.

Typical results for \( v_z \) are shown in Fig. 1 for \( \epsilon = -0.01 \) and \( L = 20.68 \). The boundaries at the ends of the apparatus cause noticeable vortex flow even for negative \( \epsilon \). We determined the local wavelength \( \lambda \) from the zero crossings of \( v_z \). In addition, the data over the central two vortex pairs (indicated by the horizontal bar in Fig. 1) were fitted by the equation

\[
v_z = \sum_{n=1}^{4} A_n \sin[\eta q (x - x_0)] + \delta_0,
\]

with \( q, x_0, \delta_0, \) and the \( A_n \) adjustable (the offset \( \delta_0 \) al-
flow for any slight contributions from the azimuthal velocity component. Our spatial resolution was about 0.01 mm, or roughly 0.1% of λ.

We compared the flow pattern to numerical solutions of the amplitude equation:

$$\xi_0 \frac{d^2 A}{dz^2} + \epsilon A - g A |A|^2 = 0,$$

with \( \xi_0 = 0.2693 \) and \( g = 1 \). The complex amplitude \( A(z) \) yields the stream function \( \psi = \text{Re}[A \exp(iq_0z)] \) which has the same periodicity as the velocity. Thus the local wave vector \( q(z) \) is given by

$$q(z) = q_0 + d\phi/\text{dz},$$

where \( \phi \) is the phase of

$$A(z) \equiv |A(z)| \exp[i\phi(z)].$$

The boundary conditions imposed on the magnitude of \( A \) were \( |A(0)| = |A(L)| = A_0 \), and the results were insensitive to either \( A_0 \) or \( q \). The phase \( \phi(0) \) was fixed at zero, and \( \phi(L) \) was adjusted for each \( \epsilon \) and \( L \) until the mean wavelength of the central nine vortex pairs equaled that of the physical system. Thus the system was effectively modeled as consisting of nine vortex pairs confined between the Ekman vortices.

The critical angular velocity \( \omega_c \) was determined to within ±0.05% by the following procedure. Equation (1) was used to extract \( A_1(\epsilon) \) from the stream function corresponding to the numerical solutions of Eq. (2), by fitting of the central two wavelengths. The numerical results for \( A_1(\epsilon) \) were then used to fit experimental data for \( A_1(\epsilon, q=q_c) \), over the range \(-0.02 \leq \epsilon \leq 0.06 \), with \( \omega_c \) being adjusted to optimize the fit. For our apparatus, we found \( \omega_c \) to be very weakly time dependent, presumably because of water evaporation. Aside from this small effect, \( \omega_c \) was taken as a single fixed number independent of \( \bar{q} \).

FIG. 1. Axial velocity \( v_z \) vs axial position (in units of the gap) for Taylor-vortex flow, at \( \epsilon = -0.01 \), for an aspect ratio of 20.68. Note the large amplitude of the Ekman vortices at the ends, which drive the flow even for \( \epsilon < 0 \). The vertical dashed line indicates the center of the apparatus.

FIG. 2. The difference between the local wavelength \( \lambda \) and the critical wavelength \( \lambda_c \) vs axial position for various values of \( \epsilon \), with an aspect ratio of 21.27. The solid circles in (a) show the large width of the Ekman vortices which are always present. Note how the wavelength distribution, which is quite nonuniform for \( \epsilon \leq 0 \), becomes uniform with increasing \( \epsilon \). The solid lines are the result obtained by solution of the amplitude equation discussed in the text.

For \( \epsilon \leq 0.01 \), the vortex wavelength was generally nonuniform, but it became steadily more uniform with increasing \( \epsilon \). This is shown in Fig. 2, where we plot the difference between the local wavelength \( \lambda(z) \) and the critical wavelength \( \lambda_c = 2\pi/q_c = 2.004 \) versus axial position, for three different values of \( \epsilon \). The vertical range of Fig. 2(a) has been enlarged to show the anomalous width of the Ekman vortices. Although not shown here, the amplitude \( |A(z)| \) has a minimum in the center, corresponding to the maximum in \( \lambda - \lambda_c \). With increasing \( \epsilon \), as shown by Fig. 2(c), the deviation from \( \lambda_c \) became increasingly more uniform throughout the apparatus. Such a deviation was necessary because the aspect ratio had been chosen so as to be incommensurate with nine vortex pairs at \( \lambda_c \) and one pair of Ekman vortices. Of course, the sign of the required deviation can be varied by changing \( L \). The solid lines in Fig. 2 are the local wavelength \( \lambda(z) \equiv 2\pi/q(z) \) obtained from Eqs. (2) to (4). As can be seen, the theory agrees well with experiment.

By substituting Eq. (4) into Eq. (2) one can show that the quantity \( |A(z)|^2 \delta\phi/\text{dz} \) is \( z \) independent. One manifestation of this has been shown in Fig. 2 with the greatest wavelength variation occurring where the amplitude was the smallest. Correspondingly, the amplitude \( |A(z)| \) is suppressed when \( |\delta\phi/\text{dz}| \) is large. This can be a substantial effect as shown by the data of Fig. 3, where the amplitude \( A_1 \) for the central two vor-
FIG. 3. The amplitude of the fundamental spatial Fourier component of the central two vortex pairs as a function of $\epsilon$ for aspect ratios $L = 20.58$ (open circles) and $L = 21.47$ (solid circles). The curves are labeled with the deviation of the average wave vector $\bar{q}$ from the critical value $q_c$. As may be seen, the amplitude is considerably suppressed for $\bar{q} \neq q_c$. This effect is accurately predicted by the amplitude equation, the results of which are shown as the solid curves.

The $\epsilon$ pairs is plotted versus $\epsilon$ for $L = 20.58$ (open circles) and $L = 21.47$ (solid circles). The solid lines are again the results of solving Eq. (2), with the overall scale factor relating the numerical results to the data having been optimized.

To explore more fully the usefulness of Eq. (2), we determined $q$ for the central two vortex pairs as a function of $\epsilon$, for various values of $\bar{q}$. The results are shown in Fig. 4, together with the neutral-stability$^{13}$ (dashed) and Eckhaus-stability$^{11}$ (dash-dotted) curves of the infinite system. The solid lines are the results of solving Eq. (2) [both the data for $\nu_2$ and the results for the stream function were fitted with Eq. (1) to obtain $q_2$]. In general the agreement is quite remarkable. Deviations are present only for large values of $|q - q_c|$, where they might be expected since Eq. (2) is an expansion about $q = q_c$, $\epsilon = 0$.

We also investigated aspect ratios for which the ten-pair state underwent a transition to nine or eleven pairs as shown by the data of Fig. 5. For each set of data in the figure, the lowest $\epsilon$ point is the smallest $\epsilon$ for which the ten-pair state was stable, with $\epsilon$ being decreased in the step sizes shown. As in Fig. 4, the neutral- and Eckhaus-stability limits are indicated by the dashed and dash-dotted curves, respectively. The solid lines show the results of solving Eq. (2), with $\epsilon$ being decreased in steps of 0.002. The lines terminate at the smallest $\epsilon$ for which a stable ten-pair solution was observed. The transition from ten to nine or eleven pairs was evidenced by a $2\pi$ jump in the phase difference $\phi(L) - \phi(0)$. Although the agreement between the amplitude equation and the data is very good, it begins to fail at larger values of $\epsilon$ or $|q - q_c|$ as

FIG. 4. The $\epsilon$ dependence of the local wave vector $q$ for the central two vortex pairs for various mean values of $\bar{q}$. From left to right: $2\pi N/L = 2.926, 2.954, 2.982, 3.038, 3.099, 3.129, \text{and } 3.192$, respectively. The small difference between these values and those of $q$ in the figure at large $\epsilon$ are due to the anomalous width of the Ekman vortices. The neutral- and Eckhaus-stability limits of the infinite system are indicated by the dashed and dash-dotted curves, respectively. Note that over the entire $q$ range shown, the Eckhaus instability has been completely suppressed, and that for $\epsilon \leq 0$ the local wave vector can lie well outside even the neutral-stability curve. The solid lines are the result of solving the amplitude equation, which clearly provides an excellent description of this effect.

FIG. 5. Data like those of Fig. 4, but for values of $\bar{q}$ sufficiently different from $q_c$ to result in transitions from the ten-vortex-pair state to either nine- or eleven-pair states with decreasing $\epsilon$. From left to right: $2\pi N/L = 2.528, 2.612, 2.656, 2.701, 2.748, 2.797, 2.847, 2.900, 3.209, 3.292, 3.362, 3.435, 3.512, 3.591, \text{and } 3.675$, respectively. For each sequence of data points the one at lowest $\epsilon$ shows the lowest stable flow observed when decreasing $\epsilon$ in the steps shown by the data. The solid curves are the result of solving the amplitude equation, and they terminate at the smallest $\epsilon$ for which a stable solution existed, when decreasing $\epsilon$ in steps of 0.002.
would be expected.

Figures 4 and 5 also indicate some unexpected asymmetry, with the deviations of the data from Eq. (2) more pronounced for \( q > q_c \) than for \( q < q_c \). Similar asymmetry in the ability of Eq. (2) to model the flow accurately was also observed for the \( \epsilon \) dependence of the amplitude \( A_1 \), and for the local wavelength versus position. We believe that this asymmetry may be related to the small difference between the theoretical Eckhaus boundary and the experimental results\(^{10} \) which exists for \( q > q_c \).

Several measurements were made to determine the position in the column at which a vortex pair was gained or lost upon decreasing \( \epsilon \), and this was always found to occur near the center. This was also the case for the numerical solution of Eq. (2).

The data shown in Figs. 4 and 5 demonstrate that for finite systems the Eckhaus instability is modified and can even be suppressed completely. This effect has been predicted by Kramer and Zimmermann for finite systems with periodic solutions.\(^{12} \) The increased stability limit that we observe agrees fairly well with their prediction that no Eckhaus transition should occur for \( |\bar{q} - q_c| < \pi / L \). As shown particularly in Fig. 4, for negative \( \epsilon \) the local wave vector in the cell center can extend far beyond the Eckhaus-stability boundary provided that \( |\bar{q} - q_c| \) is sufficiently small. However, if the values of \( \epsilon \) for which the Eckhaus transition was observed are plotted against \( \bar{q} \) the results are in good agreement with previous measurements of the stability boundary.\(^{10} \) We should also mention that we have found no evidence for the increased noise or time dependence observed by Park and Donnelly\(^{16} \) for aspect ratios which are not commensurate with \( q = q_c \).

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